ELEMENTARY MATHEMATICS

SELECTED TOPICS AND PROBLEM SOLVING

G. DOROFEEV, M. POTAPOV,
N. ROZOV

ELEMENTARY MATHEMATICS

SELECTED TOPICS AND PROBLEM SOLVING

G. DOROFEEV

M. POTAPOV

N. ROZOV

TRANSLATED FROM THE RUSSIAN
BY
GEORGE YANKOVSKY

MIR PUBLISHERS
MOSCOW

Fist published 1973
Second printing 1980

На английском языке

ИЗДАТЕЛЬСТВО • МИР •

Г. В. Дорофеев, М. К. Потапо
Н. Х. Розов

Пособие по математике
для поступающих в вузы

Избранные вопросы
элементарной
математики

Издательство
«Наука»
Москва

CONTENTS

INTRODUCTION

Mathematics has long since become the basic tool of physics and technology. In recent years, mathematical methods of investigation have made deep inroads into such fields of knowledge as chemistry, biology, economics, geology, linguistics, medicine, teaching, psychology, archeology, law and military affairs.

The school course of elementary mathematics is fundamental to all mathematical knowledge, and without a firm grasp of this basic course there can be no question of mastering the higher divisions of the subject, of applying mathematics in one's practical scientific and technological work.

It is a truism that mathematical knowledge does not merely amount to memorizing a large number of formulas; problem solving lies at the very heart of mathematics. But to solve a problem does not only mean to perform a certain number of manipulations. The most important thing is that the solution be complete and logically flawless. This is the main stumbling block to the student, for it is much easier to remember a certain number of formulations or to work through specific procedures than it is to comprehend the essence of the matter at hand.

The purpose of this text is to help the student think through the logical processes of a solution, and to teach the student to ask himself why a certain thing is being done and to be able to answer that question. It is vitally important that the student be able at every stage in a solution to realize what has been done and what there is left to do. In short, the authors have made an attempt to show the student how to solve problems *properly*.

This approach has left its imprint on the text. The authors do not always give the best or shortest solutions, in contrast to what an experienced mathematician usually does. They strive to view the problem at hand through the eyes of the student who is not experienced, does not have at his disposal ingenious techniques, devices or special methods of solution; they seek the solution in what would appear to be the most natural way for the average student. The main thing is that such a solution is carried through with extreme logical care and is made as rigorous as possible.

The reader may find that certain simple examples are analyzed in far too great an amount of detail. But do not hurry to criticize this approach, for what appears simple may merely be something that has not been studied in sufficient depth. Also, not all solutions are given with full details. It is the hope of the authors that this text will not only be read but studied with pencil and paper in hand. A good deal is left up to the student to think through by himself. This pertains to some parts of the theory and certain stages of problem solving.

It must be emphasized that this book is not an ordinary textbook but one in which certain carefully selected topics of theory and an abundant amount of problem solving will enable the student to expand and deepen his knowledge of the school course of elementary mathematics and enable him better to begin the study of higher mathematics in higher educational institutions. The topics chosen here for detailed discussion are those that usually cause the most trouble or do not, for a variety of reasons, receive the attention they deserve. The most complicated and important parts of elementary mathematics are analyzed and illustrated in detailed problem solving and subsequent discussion. Particular attention is paid to analyzing typical mistakes of the student.

Another point to bear in mind is that the authors consider only the more traditional topics of elementary mathematics. They do not use methods of analytic geometry or differential and integral calculus; in geometry, axiomatics is not dwelt on, nor is the terminology of set theory made much use of.

This textbook is supplied with a large number of problems in the form of exercises appended to each section. The answers are given at the end of the book.

This book is aimed at a broad range of readers, from students of secondary school to students of teachers' colleges and universities, and mathematics instructors in secondary and higher educational institutions. It can also be used in self-instruction as a supplement to any standard textbook.

G. Dorofeev, M. Potapov, N. Rozov

Chapter 1 ARITHMETIC AND ALGEBRA

1.1 General remarks on arithmetic and algebra

Of fundamental importance to the student is the fact that all the concepts he employs in his mathematical discourse must be rigorously *defined*, the only exception being, of course, such starting terms as natural number, equation, point, line, plane, and the like. The requisite definitions are of course given in any textbook, but the student becomes accustomed so soon to using these concepts in solving problems that he feels more and more inclined (without always realizing it) to regard the initial notions as intuitively clear and not in any need of being defined.

The student of mathematics must at all times have a clear-cut understanding of all fundamental mathematical concepts (we will return to this subject in Secs. 2.1 and 3.1).

Also important, besides definitions, are mathematical conventions involving the formation of an entity or of a relation between entities (indicated by a special symbol). These conventions serve essentially as a *definition of the symbol* and must be memorized. For example, the plus (+) sign is used to indicate the sum of two numbers, the symbol a^2 stands for the square of the number a, which is to say the product $a \cdot a$; the fact that a is less than b, that is, the number $a - b$ is negative, is written conventionally with the aid of the $<$ sign as $a < b$.

The student will also recall the signs of weak inequalities: $\leqslant$ (less than or equal to) and $\geqslant$ (greater than or equal to). The student usually finds no difficulty when using them in formal transformations, but examinations have shown that many students do not fully comprehend their meaning.

To illustrate, a frequent answer to: "*Is the inequality* $2 \leqslant 3$ *true*?" is "No, since the number 2 is less than 3". Or, say, "*Is the inequality* $3 \leqslant 3$ *true*?" the answer is often "No, since 3 is equal to 3". Nevertheless, students who answer in this fashion are often found to write the result of a problem as $x \leqslant 3$. Yet their understanding of the sign $\leqslant$ between concrete numbers signifies that not a single specific number can be substituted in place of x in the inequality $x \leqslant 3$, which is to say that the sign $\leqslant$ cannot be used to relate any numbers whatsoever.

Actually, the situation is this: by *definition* of the sign $\leqslant$, the *inequality* $a \leqslant b$ *is considered to be true when* $a < b$ *and also when* $a = b$. Thus, the inequality $2 \leqslant 3$ *is true* because 2 *is less* than 3, and the inequality $3 \leqslant 3$ *is true* because 3 *is equal* to 3.

From this definition of the sign $\leqslant$ it follows that the inequality $a \leqslant b$ is not true only when $a > b$. For this reason, the sign $\leqslant$ may be read not only as "less than or equal to" but also as "not greater than". Thus, the inequalities $2 \leqslant 3$ and $3 \leqslant 3$ are read, respectively, as "2 is not greater than 3" and "3 is not greater than 3".

The same applies to the sign $\geqslant$, which can be read both as "greater than or equal to" and as "not less than". By *definition* of the sign $\geqslant$, the *inequality* $a \geqslant b$ *is valid if* $a > b$ *or if* $a = b$; it is not valid only if $a < b$.

Almost every student knows that the function $y = 2^x$ is defined for all real x and can readily draw the graph of the function. However, $2^{\sqrt{3}}$ is often a riddle to the student. The best he can usually do is to indicate how one should give an approximate *computation* of the number. But where is the logic? How can you expect to give an approximate computation of a number without knowing its definition?

To be able to state what the number $2^{\sqrt{3}}$ represents, one has to recall the special definition for a number raised to an irrational power, and of course it is necessary to recall the other definitions of powers having natural exponents (a zero, rational or negative exponent). Note that the general definition of a power with a natural exponent n is inapplicable when $n = 1$ since a product involving a single factor is meaningless. For this reason, the equation $a^1 = a$ is the *definition* of the first power of a number. In the very same fashion, the zero power ($a^0 = 1$) is introduced as a *definition*.

Now let us find out *why the equation*

$$(\sqrt[3]{a})^3 = a \tag{1}$$

holds true. Students often prove this by manipulating the left-hand member. This is of course permissible, but it simply indicates that the rules for handling radicals have displaced in the mind of the student the definition of a radical. Indeed, how does one define the cube root of a number? By convention, the cube root of a number a is that number whose cube is equal to a. The cube root of a number a is conventionally denoted by the symbol $\sqrt[3]{a}$. Thus, equation (1) is merely the formula for the definition of a cube root with regard for the convention concerning the meaning of the symbol $\sqrt[3]{\ }$.

The course of algebra includes a considerable number of propositions (assertions). The view is rather widely held that in geometry one has to reason rigorously and there are theorems which require careful proofs with the use of definitions but that in algebra there is only one theorem (Viète's theorem),* and the rest is just verbal formulations and formulas. This is not so in the least. Even the formula for the square of a sum is a *theorem*. The properties of the logarithmic function constitute several theorems. As in geometry, every theorem of algebra must be proved, and all the initial concepts must be defined.

Experience shows that the more ordinary an algebraic statement is and the more often it is used in problem solving, the more frequently the student forgets that he should be able not only to state it properly and employ it, but also to *prove* it. At all times, particular attention must be paid to the ability of the student to justify (substantiate) statements, particularly those which appear to be "self-evident".

All students are familiar with the formula for solving quadratic equations, but not so many know its derivation. The same difficulties are encountered when dealing with *theorems involving the solution of quadratic inequalities*. Even if the student obtains correct solutions of such inequalities, he is frequently not able to explain *why*, for instance, a quadratic trinomial with positive leading coefficient is positive outside the interval between the roots if the latter are real, and is positive for arbitrary x if the roots are imaginary.

Yet rigorous proofs of the theorems dealing with the sign of a quadratic trinomial are simple in the extreme.

If the quadratic ax^2+bx+c, $a\neq 0$, has real roots x_1 and x_2 (which means its discriminant is positive), then it can be factored:

$$ax^2+bx+c=a(x-x_1)(x-x_2) \tag{2}$$

It is thus evident that for any x exceeding the larger root, both factors in parentheses, that is $(x-x_1)$ and $(x-x_2)$, are positive, and for any x less than the smaller root, they are negative, which means that in both cases their product $(x-x_1)(x-x_2)$ is positive and therefore the right member of (2) has the same sign as the number a. However, if x lies in the interval between the roots x_1 and x_2, then one of the parentheses in (2) is positive and the other one is negative. And so the sign of the product in the right member of (2) is opposite that of a.

We have thus proved the following **theorem**: *the value of a quadratic trinomial ax^2+bx+c with positive discriminant* ($b^2-4ac>0$) *has for any x outside the interval between the roots of the quadratic a sign that coincides with the sign of the coefficient a, and is of opposite sign for any x inside the interval between the roots.* **

* Viète's theorem states that the sum of the roots of a quadratic equation is equal to the coefficient (with sign reversed) of the unknown to the first power, and the product of the roots is equal to the constant term.

** The student himself can state and prove the theorem referring to the case when the quadratic trinomial ax^2+bx+c has equal roots, i.e. when its discriminant is zero: $b^2-4ac=0$.

There is also another **theorem** that is valid: *the value of a quadratic* ax^2+bx+c *with negative discriminant* ($b^2-4ac<0$) *has for any* x *a sign coincident with the sign of the coefficient* a.

To prove this theorem, isolate a perfect square:

$$ax^2+bx+c=a\left[\left(x+\frac{b}{2a}\right)^2+\frac{4ac-b^2}{4a^2}\right] \tag{3}$$

Since the discriminant $b^2-4ac<0$ (it will be recalled that in this case the quadratic has imaginary roots), it is evident that the expression in square brackets is positive for any value of x, and the product in the right-hand member of (3) is, for any x, of the same sign as the number a.

The student is often surprised to encounter difficulties when dealing with **biquadratic equations.** There would seem to be no difficulties, since *any biquadratic equation* $ax^4+bx^2+c=0$ *can be reduced to a quadratic equation by the standard substitution* $x^2=y$. But suppose that the resulting quadratic has imaginary roots y_1 and y_2. Then determining x requires taking the square root of a complex number. In itself this is not so complicated and appropriate formulas are given in the standard textbooks. However, this may be avoided altogether if one does not resort to the standard substitution but factors the left-hand member by means of a special transformation.

This transformation consists in isolating a perfect square in the trinomial ax^4+bx^2+c and gives a valid result only when the quadratic equation $ay^2+by+c=0$ has **imaginary** roots.

However, in this case the perfect square is isolated in a somewhat different fashion than ordinarily: namely, group together the highest-degree term and the constant term, and then take their sum and complete the square.

Suppose we have an equation like $x^4+bx^2+c=0$ (for the sake of simplicity, we set $a=1$, which can always be done readily), and the equation $y^2+by+c=0$ has imaginary roots. This condition means that the discriminant $D=b^2-4c<0$, that is, $b<4c$, whence it is clear that $c>0$ and $|b|<2\sqrt{c}$, that is $b<2\sqrt{c}$. We can therefore perfom the following manipulations:

$$x^4+bx^2+c=(x^4+c)+bx^2=(x^4+2\sqrt{c}\,x^2+c)-(2\sqrt{c}-b)\,x^2$$
$$=(x^2+\sqrt{c})^2-(2\sqrt{c}-b)\,x^2$$
$$=\left(x^2+x\sqrt{2\sqrt{c}-b}+\sqrt{c}\right)\left(x^2-x\sqrt{2\sqrt{c}-b}+\sqrt{c}\right)$$

The solution of the given biquadratic equation now reduces to that of two quadratic equations with real coefficients.

These rather involved formulas need not be memorized of course; it is much better to isolate a perfect square in each given instance. To illustrate, let us solve the equation

$$2x^4+2x^2+3=0$$

We first reduce the equation to $x^4+x^2+{}^3/_2=0$. Its discriminant is equal to $1^2-4\cdot{}^3/_2=-5<0$, and so, applying the foregoing method, we obtain

$$\begin{aligned}x^4+x^2+{}^3/_2&=(x^4+2\sqrt{{}^3/_2}x^2+{}^3/_2)-(2\sqrt{{}^3/_2}-1)x^2\\&=(x^2+\sqrt{{}^3/_2})^2-(\sqrt{6}-1)x^2\\&=\left(x^2+x\sqrt{\sqrt{6}-1}+\sqrt{{}^3/_2}\right)\left(x^2-x\sqrt{\sqrt{6}-1}+\sqrt{{}^3/_2}\right)\end{aligned}$$

We can now solve the quadratics without any fear of complicated radicals. The first equation

$$x^2+x\sqrt{\sqrt{6}-1}+\sqrt{{}^3/_2}=0$$

has a negative discriminant: $D=\left(\sqrt{\sqrt{6}-1}\right)^2-4\sqrt{{}^3/_2}=-1-\sqrt{6}$, and, consequently, its roots

$$x_{1,2}=-\frac{\sqrt{\sqrt{6}-1}}{2}\pm i\frac{\sqrt{\sqrt{6}+1}}{2}$$

Similarly we find the roots of the second equation:

$$x^2-x\sqrt{\sqrt{6}-1}+\sqrt{{}^3/_2}=0$$

They are

$$x_{3,4}=\frac{\sqrt{\sqrt{6}-1}}{2}\pm i\frac{\sqrt{\sqrt{6}+1}}{2}$$

Two-term equations of the sixth degree ($x^6+a^6=0$) likewise reduce to the solution of this type of biquadratic equation (expand the left-hand member as a sum of cubes and apply the technique described above).

A few words are in order concerning the statements of a number of definitions and theorems. Textbooks frequently state definitions and theorems verbally without much use of convenient literal notation. Occasionally, this is justified, but very often it simply makes for hard-to-digest formulations. For instance, instead of writing "the square of the sum of any two numbers is equal to the sum of the squares of the numbers plus two times their product," one could more simply write: "for any numbers a and b we have $(a+b)^2=a^2+2ab+b^2$." A logarithm is conveniently defined as "a number x is the logarithm of a number N to the base a ($a>0$, $a\neq 1$) if $a^x=N$."

It is important to develop the habit of converting verbal statements into formula statements, and vice versa, for this is precisely what is ordinarily required when proving theorems. For example, to prove that "the logarithms of numbers exceeding unity to a base exceeding unity are positive," we must first introduce the designations: let the base be $a>1$, the number $x>1$, and let $y=\log_a x$; then establish that the number $y>0$. Rephrasing of this nature can also involve the necessity of using a definition. Thus, before proving the assertion that "for $a>1$ the function $y=\log_a x$ increases," one has to recall what an increasing function is, and then the proof begins thus: "Let $a>1$, and let x_1 and x_2 be positive numbers, $x_1<x_2$; we will prove that $\log_a x_1<\log_a x_2$."

It is not always properly understood that certain formula-type statements make use of symbols of certain concepts.

It is precisely this that explains why formula (1) is not readily recognized as the definition of a cube root written as the symbol $\sqrt[3]{\ }$, and that the equation $a^{\log_a N}=N\ (N>0,\ a>0,\ a\neq 1)$ is a symbolic notation of the "customary" verbal definition of a logarithm which employs the convention of denoting the logarithm of a number N to a base a in the form of $\log_a N$.

Exercises

1. What is (a) a periodic decimal fraction, (b) $a^{2/3}$, (c) a quadratic equation, (d) $\sqrt{11}$, (e) the modulus (absolute value) of a complex number, (f) $a>b$, (g) the sum of a nonterminating decreasing geometric progression?

2. State which of the following is a definition, an axiom or a theorem: (a) an equation is unaltered if both members are multiplied by the same number, (b) the modulus of any number is nonnegative, (c) $a^{1/3}=\sqrt[3]{a}$, (d) the graph of the function $y=-3x$ passes through the origin of coordinates.

3. Is the following equation always valid: $\sqrt{a}\sqrt{b}=\sqrt{ab}$?

4. If the discriminant of a quadratic equation is positive, then the equation has two distinct real roots. State the converse theorem, the inverse and the contrapositive. Which of these theorems are valid?

5. Prove that if the roots of a quadratic equation are imaginary, then the discriminant is negative.

6. Using formulas, state the condition that at least one of the numbers $a_1, \ldots, a_n$ is equal to the number α.

7. Use a single equation to denote that at least two of the numbers a, b, c are equal to zero.

8. What can be said about the numbers a and b if $1/a<1/b$? From what properties of the function $y=1/x$ can we obtain an answer to this question?

9. Using mathematical relations, state the assertion that the function $y=3x-x^2$ increases when the argument varies in the interval from -1 to $+1$.

10. Is the condition that the sum of the digits of a number is divisible by 3 a necessary, sufficient or necessary and sufficient condition for the number to be divisible by 12?

1.2 Integers, rational numbers, irrational numbers

Problems involving various parts of arithmetic often give trouble. This is frequently due to the fact that arithmetic is studied in the junior forms where many results are given without proof, and the material is actually never taken up again. Yet this does not in the least diminish the significance of such sections of arithmetic as

the divisibility of the natural numbers, the propert
the theory of proportions, etc.

The senior student must know the statements of these results and should also be able to prove them (say, to derive a given criterion of divisibility).

To illustrate, let us prove the *criterion for divisibility by* 9. Given a natural number $N=\overline{a_n a_{n-1} \ldots a_2 a_1 a_0}$. Here, the symbol $\overline{a_n a_{n-1} \ldots a_1 a_0}$ (where the bar on top means that the digits are not to be thought of as a product of the numbers $a_n, \ldots, a_0$) denotes an $(n+1)$-digit number, where $a_n, a_{n-1}, \ldots, a_1, a_0$ are the digits of the appropriate orders of the number* (so that $1 \leqslant a_n \leqslant 9$, $0 \leqslant a_{n-1} \leqslant 9, \ldots, 0 \leqslant a_1 \leqslant 9$, $0 \leqslant a_0 \leqslant 9$). We have to prove two assertions: (a) if the sum of the digits $a_n+a_{n-1}+\ldots+a_1+a_0$ of the number N is divisible by 9, then the number N itself is divisible by 9; (b) if the number N is divisible by 9, then the sum of its digits is divisible by 9.

In accord with the positional principle of the decimal number system, we have

$$\overline{a_n a_{n-1} \ldots a_2 a_1 a_0} = a_n \cdot 10^n + a_{n-1} \cdot 10^{n-1} + \ldots + a_2 \cdot 10^2 + a_1 \cdot 10 + a_0$$

Since $10^k = \underbrace{\overline{99 \ldots 9}}_{k \text{ times}} + 1$ for any natural $k \geqslant 1$, we get

$$N = [a_n \underbrace{\overline{99 \ldots 9}}_{n \text{ times}} + a_{n-1} \underbrace{\overline{99 \ldots 9}}_{n-1 \text{ times}} + \ldots + a_2 \cdot \overline{99} + a_1 \cdot 9] + (a_n + a_{n-1} + \ldots + a_2 + a_1 + a_0) \quad (1)$$

It is obvious that the number in square brackets is divisible by 9, for it is a sum of n terms, each of which is divisible by 9. If the sum $a_n+\ldots+a_1+a_0$ is divisible by 9, then from (1) it is clear that the number N is also divisible by 9. The proof of Assertion (a) is complete. Assertion (b) likewise follows from a consideration of (1): if the left member (the number N) is divisible by 9 and since the first summand of the right member (the number in square brackets) is divisible by 9, it follows that the second summand (the sum of the digits of N) must be divisible by 9.

In the solution of problems, various arithmetical facts are sometimes useful. We shall now review a number of them using literal symbolism.

If we have two integers** a and b, $b>0$, then there is a unique integer q and a unique integer r, $0 \leqslant r < b$, such that

$$a = bq + r \quad (2)$$

* It is natural to regard the highest-order digit as nonzero.

** Recall that the numbers 1, 2, 3, ... are called *natural numbers* (the positive integers), and the numbers —2, —1, 0, 1, 2, ... are the *integers* (whole numbers). It is convenient to write the set of integers as 0, ± 1, ± 2, ...

Equation (2) is simply the *division of a number a by a number b with a remainder*. In particular, from (2) it is clear that any even number is of the form $2k$, where k is an integer, and any odd number may be represented as $2n+1$, where n is an integer.

If we have a natural number N exceeding unity, and if $N=n_1^{\alpha_1} \dots n_k^{\alpha_k}$ is the decomposition of this number into prime factors (here, $n_1, \dots, n_k$ are distinct prime divisors of N, and $\alpha_1, \dots, \alpha_k$ represent the number of their repetitions in the decomposition of N), then any divisor of N is of the form $D=n_1^{\beta_1} \dots n_k^{\beta_k}$ where $0 \leqslant \beta_1 \leqslant \alpha_1, \dots, 0 \leqslant \beta_k \leqslant \alpha_k$.

If we have natural numbers $a_1, \dots, a_n$, then their *common divisor* is a natural number which exactly divides each of the numbers $a_1, \dots, a_n$. The largest of these common divisors of the numbers $a_1, \dots, a_n$ is termed the greatest common divisor. If the greatest common divisor is equal to 1, then the numbers $a_1, \dots, a_n$ are *relatively prime* (coprime).

If a natural number N is divisible by each of two relatively prime integers a_1, a_2, then N is also divisible by the product a_1a_2 of these integers.* Furthermore, if the product NM of natural numbers N and M is exactly divisible by a natural number D and if M and D are relatively prime, then N is divisible by D.

Finally, it is well to recall the following property: one of a sequence of n integers $k+1, k+2, \dots, k+n$, where k is an arbitrary integer, is definitely divisible by n.

Let us consider some examples of the use of the properties of integers in solving problems which involve divisibility.

1. *Prove that for an arbitrary even n the number $N=n^3+20n$ is divisible by* 48.

Quite naturally, a direct verification of the fact that the assertion holds true for $n=2, 4, 6, \dots$ does not solve the problem since we are not able to run through all the even numbers. Hence, we have to give a proof that will hold true for any even n.

An even number n can be written in the form $n=2k$, where k is an integer; therefore $N=8k(k^2+5)$. If we demonstrate that for *any* integer k the number $k(k^2+5)$ is divisible by 6, it will be clear that N is divisible by 48.

We perform the following obvious transformation:

$$k(k^2+5)=k(k^2-1+6)=(k-1)k(k+1)+6k \qquad (3)$$

We see that the second summand in the right member of (3) is divisible by 6. Now the first summand on the right is a product of *three* successive integers, and for this reason one of them is definitely divisible

* It is easy to see that if a_1 and a_2 are not relatively prime, then the number N is not necessarily divisible by the product a_1a_2 (give an example!).