History of Mathematics

SOURCES

Volume 39

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RICHARD DEDEKIND

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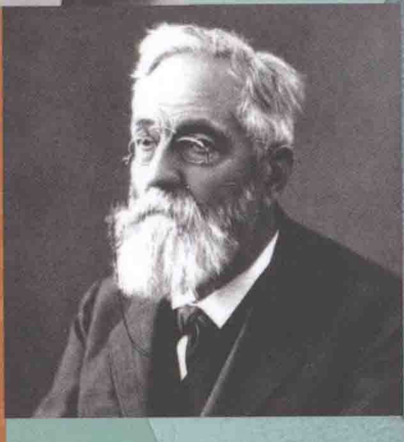

American Mathematical Society

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2010 *Mathematics Subject Classification.* Primary 01-02, 01A55.

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Library of Congress Cataloging-in-Publication Data

Dedekind, Richard, 1831–1916.
[Theorie der algebraischen Functionen einer Veränderlichen. English]
Theory of algebraic functions of one variable / Richard Dedekind and Heinrich Weber ; Translated and introduced by John Stillwell.
p. cm. — (History of mathematics ; v. 39)
Includes bibliographical references and index.
ISBN 978-0-8218-8330-3 (alk. paper)
1. Algebraic functions. 2. Geometry, Algebraic. I. Weber, Heinrich, 1842–1913. II. Title.

QA341.D4313 2012
512.7′3—dc23

2012011949

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THEORY OF ALGEBRAIC FUNCTIONS OF ONE VARIABLE

Preface

Dedekind and Weber's 1882 paper on algebraic functions of one variable is one of the most important papers in the history of algebraic geometry. It changed the direction of the subject, and established its foundations, by introducing methods from algebraic number theory. Specifically, they used rings and ideals to give rigorous proofs of results previously obtained, in nonrigorous fashion, with the help of analysis and topology. Also, by importing ideas from number theory, the paper revealed the deep analogy between number fields and function fields—an analogy that continues to benefit both number theory and geometry today.

The influence of the paper is obvious in 20th-century algebraic geometry, where the role of arithmetic/algebraic methods has increased enormously in both scope and sophistication. But, as the sophistication of algebraic geometry has increased, so has its detachment from its origins. While the Dedekind-Weber paper continues to be cited, I venture to guess that few modern algebraic geometers are familiar with its contents. There are a few useful commentaries on the paper, but those that I know seem to focus on a few of the concepts used by Dedekind and Weber, while ignoring others. And, of course, fewer mathematicians today are able to read the language in which the paper was written (and I don't mean only the German language, but also the mathematical language of the 1880s).

I therefore believe that it is time for an English edition of the paper, with commentary to assist the modern reader. My commentary takes the form of a Translator's Introduction, which lays out the historical background to Dedekind and Weber's work, plus section-by-section comments and footnotes inserted in the translation itself. The comments attempt to guide the reader through the original text, which is somewhat terse and unmotivated, and the footnotes address specific details such as nonstandard terminology. The historical background is far richer than could be guessed from the Dedekind-Weber paper itself, including such things as Abel's results in integral calculus, Riemann's revolutionary approach to complex analysis and his discoveries in surface topology, and developments in number theory from Euler to Dedekind. The background is indeed richer than some readers may care to digest, but it is a background against which the clarity and simplicity of the Dedekind-Weber theory looks all the more impressive.

I hope that this edition will be of interest to several classes of readers: historians of mathematics who seek an annotated edition of one of the classics, mathematicians interested in history who would like to know where modern algebraic geometry came from, students of algebraic geometry who seek motivation for the concepts they are studying, and perhaps even algebraic geometers who have not had time to catch up with the origins of their discipline. (It seems to an outsider that just the modern literature on algebraic geometry would take more than a lifetime to absorb.)

This translation was originally written in the 1990s, but in 2011 I was motivated to revise it and write an introduction in order to prepare for a summer school presentation on ideal elements in mathematics. I have also compiled a bibliography and index. The bibliography is mainly for the Translator's Introduction, but it is occasionally referred to in the commentary on the translation, so I have placed it after the translation.

The summer school, PhilMath Intersem, was organized by Mic Detlefsen, and held in Paris and Nancy in June 2011. I thank Mic for inviting me and for support during the summer school. I also thank Monash University and the University of San Francisco for their support while I was researching this topic and writing it up. Anonymous reviewers from the AMS have been very helpful with some technical details of the translation, and I also thank Natalya Pluzhnikov for copyediting. Finally, I thank my colleague Tristan Needham, my wife Elaine, and son Robert for reading the manuscript and saving me from some embarrassing errors.

John Stillwell

South Melbourne, 1 May 2012

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Translator's Introduction

1. Overview

> Modern algebraic geometry has deservedly been considered for a long time as an exceedingly complex part of mathematics, drawing practically on every part to build up its concepts and methods and increasingly becoming an indispensable tool in many seemingly remote theories. It shares with number theory the distinction of having one of the longest and most intricate histories among all branches of our science, of having always attracted the efforts of the best mathematicians in each generation, and of still being one of the most active areas of research.
>
> Dieudonné (1972), p. 827.

> It seems to me that, in the spirit of the biogenetic law, the student who repeats in miniature the evolution of algebraic geometry will grasp the logic of the subject more clearly.
>
> Shafarevich (1994), p. vii.

Richard Dedekind and Heinrich Weber first worked together in 1874, as co-editors of Riemann's collected works. Weber was called into this project as a replacement for Clebsch, who had died unexpectedly of diptheria, and his expertise in mathematical physics complemented Dedekind's expertise in pure algebra and analysis. The fruit of this collaboration was their joint paper, Dedekind and Weber (1882), a ground-breaking contribution to the understanding and advancement of Riemann's ideas. *Theorie der algebraischen Functionen der einer Veränderlichen* (theory of algebraic functions of one variable) revolutionized algebraic geometry by introducing methods of algebraic number theory into the subject. This made possible the first rigorous proofs of theorems discovered with the help of physical intuition, and opened the way to an extension of algebraic-geometric concepts from the complex numbers to arbitrary fields.

In a sense, the paper is a sequel to Dedekind (1877), a long paper in which Dedekind expounded his theory of ideals and their applications to number theory. However, Dedekind and Weber give a self-contained exposition of their theory, which is at some points simpler than the ideal theory for algebraic numbers.

Like Dedekind (1877), the Dedekind-Weber paper starts with the concept of field, but this time it is a field of *functions*, the "algebraic functions of one variable." Following the example of number theory, they distinguish the ring of *integers* of this field, then the *primes*, and finally the *ideals*. As in number theory, it turns out that ideals are crucial to complete the analogy with the traditional arithmetic of integers. However, in the context of algebraic functions, ideals prove to be important in other ways, and indeed a more general idea that they call "polygons" is needed.

To show the value of these new ideas, Dedekind and Weber gave new proofs of two great theorems: *Abel's theorem* of Abel (1841) and the *Riemann-Roch theorem* of Riemann (1857) and Roch (1864). These theorems are as timely today as they were in 1882, but they require some introduction, which Dedekind and Weber do not supply. I would therefore like to present some historical background to these theorems, and to the theory of algebraic functions itself, with copious examples. In some ways, this introduction is a sequel to my introduction to Dedekind (1877), though I will recapitulate some points to keep it self-contained.

In preparing this material I have been greatly assisted by the first 35 pages of Dieudonné (1985), an extraordinarily rich and insightful account of the development of algebraic geometry up to the Dedekind-Weber paper, and Koch (1991), which places this development against the general background of 19th-century mathematics. Another helpful overview is the chapter by Geyer in Dedekind et al. (1981). As will become apparent, much of the algebra in modern algebraic geometry arose from problems in classical analysis, particularly the integral calculus. The first such result was the fundamental theorem of algebra, originally motivated by the desire to factorize polynomials for the purpose of integrating rational functions.

2. From Calculus to Abel's Theory of Algebraic Curves

> What a discovery is Abel's generalization of Euler's integral! I have never seen such a thing! But how can it be that this discovery, which could be the most important made in the mathematics of this century, and which was communicated to your Academy two years ago, has escaped the attention of you and your colleagues?
>
> Jacobi (1829) letter to Legendre, 14 March 1829.

When calculus was developed in the 17th century, the first really hard problems were problems of integration. This was especially true of the Leibniz approach, which sought integrals in "closed form," that is, in terms of functions from the small class known as "elementary." These are the algebraic functions, together with functions arising from them by composition with the exponential function and its relatives, the logarithm, circular functions, and inverse circular functions.

The only broad class of functions that can be integrated in Leibniz's sense are the rational functions, that is, the functions of the form $r(x) = p(x)/q(x)$, where p and q are polynomials. Any rational function can be integrated because the denominator $q(x)$ may be split into linear factors $(x-a)$, by the fundamental theorem of algebra, and the quotient $p(x)/q(x)$ may then be decomposed into partial fractions of the form $x^m/(x-a)^n$, which have rational integrals in all cases except

$$\int \frac{dx}{x-a} = \log(x-a) + \text{constant}.$$

Thus the integral of a rational function is itself a rational function, with the possible exception of some terms of the form $\log(x-a)$.

(In elementary calculus courses this simple picture is confused by the presence of partial fractions such as $1/(x^2+1)$, the integral of which is usually taken to be $\tan^{-1} x + \text{constant}$. However, we have

$$\frac{1}{x^2+1} = \frac{1/2i}{x-i} - \frac{1/2i}{x+i},$$

so we can also express $\int dx/(x^2+1)$ as a sum of logarithms, namely

$$\int \frac{dx}{x^2+1} = \frac{1}{2i}\int \frac{dx}{x-i} - \frac{1}{2i}\int \frac{dx}{x+i} = \frac{1}{2i}\log(x-i) - \frac{1}{2i}\log(x+i).$$

This was first done, albeit with some confusion about the meaning of complex logarithms, by Johann Bernoulli (1702). Around 1800, when the fundamental theorem of algebra was finally proved, the meaning of complex numbers became better understood, and it became increasingly clear that they play an important role in the theory of integrals.)

When the rational functions are extended by as little as the square root function, the resulting integrals quickly fall outside the class of elementary functions. A famous example is the *lemniscatic* integral

$$\mathrm{sl}^{-1}(x) = \int_0^x \frac{dt}{\sqrt{1-t^4}},$$

so-called because it expresses the arc length of the lemniscate of Jakob Bernoulli (1694), shown in Figure 1.

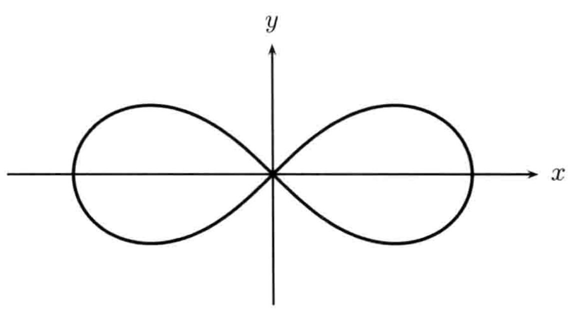

FIGURE 1. The lemniscate of Bernoulli

This curve has cartesian equation $(x^2+y^2)^2 = x^2-y^2$, and its arc length cannot be expressed in terms of elementary functions of x and y. However, Fagnano (1718) discovered that the lemniscatic integral satisfies an arc-length *doubling formula*

$$2\int_0^x \frac{dt}{\sqrt{1-t^4}} = \int_0^{2x\sqrt{1-x^4}/(1+x^4)} \frac{dt}{\sqrt{1-t^4}},$$

and Euler (1768) generalized Fagnano's formula to an arc-length *addition formula*

$$(*) \qquad \int_0^x \frac{dt}{\sqrt{1-t^4}} + \int_0^y \frac{dt}{\sqrt{1-t^4}} = \int_0^{(x\sqrt{1-y^4}+y\sqrt{1-x^4})/(1+x^2y^2)} \frac{dt}{\sqrt{1-t^4}}.$$

These results are analogous to properties of the *inverse sine integral*

$$\theta = \sin^{-1} x = \int_0^x \frac{dt}{\sqrt{1-t^2}},$$

which are derivable from basic properties of the sine and cosine functions. For example, the familiar angle-doubling formula

$$\sin 2\theta = 2\sin\theta\cos\theta = 2\sin\theta\sqrt{1-\sin^2\theta},$$

implies that

$$2\theta = \sin^{-1}(2\sin\theta\sqrt{1-\sin^2\theta}),$$

which gives the angle-doubling formula for integrals:

$$2\sin^{-1}x = 2\int_0^x \frac{dt}{\sqrt{1-t^2}} = \int_0^{2x\sqrt{1-x^2}} \frac{dt}{\sqrt{1-t^2}}.$$

And the familiar angle addition formula,

$$\begin{aligned}\sin(\theta+\varphi) &= \sin\theta\cos\varphi + \cos\theta\sin\varphi \\ &= \sin\theta\sqrt{1-\sin^2\varphi} + \sin\varphi\sqrt{1-\sin^2\theta},\end{aligned}$$

implies

$$\theta+\varphi = \sin^{-1}\left(\sin\theta\sqrt{1-\sin^2\varphi} + \sin\varphi\sqrt{1-\sin^2\theta}\right),$$

which gives the addition formula for arcsine integrals:

$$\int_0^x \frac{dt}{\sqrt{1-t^2}} + \int_0^y \frac{dt}{\sqrt{1-t^2}} = \int_0^{x\sqrt{1-y^2}+y\sqrt{1-x^2}} \frac{dt}{\sqrt{1-t^2}}.$$

Thus in both cases we find that a sum of two integrals, $\int_0^x f(t)\,dt + \int_0^y f(t)\,dt$, can be simplified to a single integral, $\int_0^z f(t)\,dt$, where z is an algebraic function of x and y.

It so happens that the integrand $1/\sqrt{1-t^2}$ of the inverse sine integral can be *rationalized* by the change of variable $t = 2s/(1+s^2)$—not so surprisingly, since the inverse sine is an elementary function—so we can eliminate the integral altogether in this case. However, in the case of the lemniscatic integral, reducing the sum of two integrals to one is the best we can do. The integrand $1/\sqrt{1-t^4}$ *cannot* be rationalized by a change of variable, and indeed Jakob Bernoulli (1704) made a remarkable attempt to prove this, using the theorem of Fermat that the equation $X^4 - Y^4 = Z^2$ has no solution in positive integers X, Y, Z. His attempt fell short, because it is not enough to know this theorem for integers X, Y, Z. But it is enough to know it for *polynomials* $X(t), Y(t), Z(t)$, and indeed polynomials behave enough like integers that Fermat's proof can be replayed for polynomials, though no one noticed this in Bernoulli's time.

Thus there is an essential difference between the ordinary sine function and the lemniscatic sine function, sl, defined as the inverse of the lemniscatic integral. Nevertheless there are enough similarities to enable the development of a theory of the lemniscatic sine function. This was begun by Gauss in 1796, and extended to a general theory of the so-called *elliptic functions* by Abel and Jacobi in the 1820s. Like the circular functions, the elliptic functions satisfy addition formulas and they are *periodic*, only more so. Just as the sine and cosine have *period* 2π, in the sense that

$$\sin(\theta+2\pi) = \sin\theta, \quad \cos(\theta+2\pi) = \cos\theta,$$

an elliptic function f has *two* periods ω_1, ω_2, in the sense that

$$f(z+\omega_1) = f(z), \quad f(z+\omega_2) = f(z).$$

The periods ω_1, ω_2 are complex numbers whose ratio is not real. For example, Gauss discovered that the two periods of sl are ϖ and $i\varpi$, where

$$\varpi = 2\int_0^1 \frac{dt}{\sqrt{1-t^4}}.$$

The double periodicity of elliptic functions was first explained by algebraic manipulation of integrals, but Riemann (1851) found a far more transparent geometric explanation (not unlike explaining the period 2π of sine and cosine by referring to the circle), which we will come to later.

The theory of elliptic functions was the first great advance in integral calculus since the integration of rational functions. Nevertheless, this theory only scratched the surface of a huge and important world of calculus: the integrals of *algebraic functions*; that is, integrals of the form

$$\int g(s,t)\,dt, \quad \text{where } s \text{ satisfies a polynomial equation} \quad P(s,t)=0.$$

The lemniscatic integral is $\int dt/s$, where $s^2 = 1-t^4$, and the general theory of elliptic functions deals with the integrals $\int dt/s$ where s^2 equals a polynomial of degree 3 or 4 in t. But what can one say, for example, about the integral

$$\int_0^x \frac{dt}{\sqrt{1-t^6}}\ ?$$

It turns out that this integral does *not* satisfy an addition formula

$$\int_0^x \frac{dt}{\sqrt{1-t^6}} + \int_0^y \frac{dt}{\sqrt{1-t^6}} = \int_0^z \frac{dt}{\sqrt{1-t^6}},$$

where z is an algebraic function of x and y. However, Abel (1841) discovered a wonderful substitute for an addition formula: *any sum of integrals,*

$$\int_0^{x_1} \frac{dt}{\sqrt{1-t^6}} + \cdots + \int_0^{x_m} \frac{dt}{\sqrt{1-t^6}}$$

is equal to the sum of two *integrals*

$$\int_0^{z_1} \frac{dt}{\sqrt{1-t^6}} + \int_0^{z_2} \frac{dt}{\sqrt{1-t^6}}, \quad \text{where } z_1, z_2 \text{ are algebraic functions of } x_1, \ldots, x_m,$$

plus some "trivial" algebraic and logarithmic terms.

This result is only an illustration of the amazingly general:

Abel's Theorem. *For any integral of the form $\int g(s,t)\,dt$, where g is a rational function and s and t are connected by a polynomial relation $P(s,t)=0$, there is a number p such that any sum of integrals*

$$\int_0^{x_1} g(s,t)\,dt + \cdots + \int_0^{x_m} g(s,t)\,dt$$

equals a sum of at most p integrals

$$\int_0^{z_1} g(s,t)\,dt + \cdots + \int_0^{z_p} g(s,t)\,dt,$$

where $z_1, \ldots, z_p$ are algebraic functions of $x_1, \ldots, x_m$, plus terms that are either rational functions or their logarithms.

The number p depends only on the polynomial P. It was later called the *genus* of the curve defined by $P(s,t)=0$, and it too found a natural geometric interpretation in Riemann (1851), as we will see in the next section. In particular, the curve $s^2 = 1 - t^4$ that yields the lemniscatic integral has genus 1, because any sum of lemniscatic integrals reduces to one integral by repeated application of the addition formula (*). More generally, any *elliptic curve*[1] $s^2 = q(t)$, where $q(t)$ is of degree 3 or 4 without repeated roots, is of genus 1, because there is an addition formula for the corresponding integral $\int dt/s$.

Finally, any curve $s = P(w)$, $t = Q(w)$ parameterized by rational functions P and Q is of genus zero, because the corresponding integral $\int g(s,t)\,dt$ is the integral of the rational function $g(P(w), Q(w))Q'(w)$.

Abel submitted his paper to Cauchy in 1826 but, due to inattention by the mathematicians of the Paris Academy, it was not published at the time. It was noticed by Jacobi, however, who in 1829 wrote the letter to Legendre quoted at the beginning of this section. Even the intervention of Jacobi failed to wake up the Academicians, and Abel's paper did not appear until 1841, long after Abel had died. There is a further excruciating twist to this story of neglected genius. The other mathematician notoriously ignored by the Paris Academy, Evariste Galois, also seems to have discovered Abel's theorem, independently of Abel, but some years later. It is mentioned in his letter to Auguste Chevalier, Galois (1846), written on the night before his death in 1832. He states the theorem without proof, but with some additional remarks that suggest that he already had some of the ideas developed by Riemann 20 years later.

3. Riemann's Theory of Algebraic Curves

> It is quite a paradox that in the work of this prodigious genius, out of which algebraic geometry emerges entirely regenerated, there is almost no mention of algebraic curve; it is from his theory of algebraic *functions* and their integrals that all of the birational geometry of the nineteenth and the beginning of the twentieth century issues.
>
> Dieudonné (1985), p. 18.

In the 1850s, two papers by Bernhard Riemann[2] completely changed the face of complex analysis and algebraic geometry. Riemann (1851) and Riemann (1857) viewed algebraic curves in a new way, as what we now call *Riemann surfaces.* In retrospect, this development seems unsurprising and even inevitable. Since around 1800, mathematicians had become used to the idea that the complex "line" $\mathbb{C}$ was geometrically a plane, so the idea that a complex "curve" should be some kind of surface was just over the horizon. Nevertheless, Riemann's description of these surfaces was received skeptically by most of his contemporaries. The underlying topological ideas, though very intuitive and persuasive, did not yet have a rigorous foundation. And, to make matters worse, Riemann made connections between topology and analysis by appealing to physics. Then, as now, this was considered mathematically dubious.

---

[1] The name "elliptic" became attached to the curves of genus 1 because the corresponding integrals ("elliptic integrals") include the integral for the arc length of the ellipse. Unfortunately, the ellipse itself has genus 0, and hence is *not* an elliptic curve.

[2] Page numbers in references to these papers in this Introduction refer to the original papers. However, many readers will find it more convenient to consult the English translation of Riemann's works, Riemann (2004). To make this easier to do, I also give section numbers, which are the same in the original papers and in the translation.

But if Riemann's proofs were not rigorous, his results were so stunning that they demanded explanation, and this became the task of later mathematicians, among them Dedekind and Weber.

Today, the necessary foundations of topology and analysis have been constructed, so we have the luxury of describing Riemann's ideas in informal terms similar to his own. I think that it is useful to do so, because some of the algebraic concepts devised by Dedekind and Weber are scarcely comprehensible if one has not seen the topological concepts they replace. In particular, I doubt that readers should be confronted with the "ramification ideal" before they have seen a picture of "ramification," or "branching." Such a picture was given in Neumann (1865), the first textbook on Riemann's theory (Figure 2).

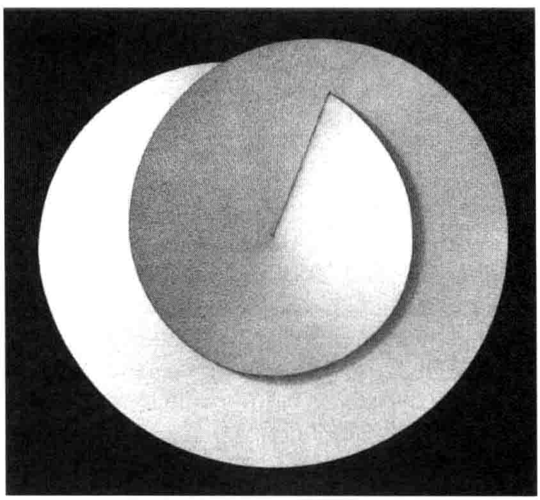

FIGURE 2. Neumann's picture of a branch point

This picture springs to mind when one attempts to visualize the curve $y^2 = x$ for *complex* variables x and y or, equivalently, the "two-valued function" $y = \pm\sqrt{x}$. Riemann imagined the two values $+\sqrt{x}$ and $-\sqrt{x}$ lying above x on a *two-sheeted covering* of the plane $\mathbb{C}$, as shown in Figure 3. Notice that, as x moves continuously once around a circle, the corresponding point $\sqrt{x}$ moves continuously around the lower sheet, then the upper sheet, of the two-sheeted covering, eventually taking the value $-\sqrt{x}$ that also lies above x. Thus the function $\sqrt{x}$ becomes "single-valued" on the covering surface.

The point $x = 0$ at which the two sheets fuse is called a *branch point* or *ramification point* of the covering, because one used to speak of the "branches of the multi-valued function"—in this case the two "branches" are $\sqrt{x}$ and $-\sqrt{x}$. The awkward feature of the picture—that the two sheets appear to pass through each other—is a result of representing the relation $y = x^2$ in three dimensions, one fewer than the four dimensions it really requires. One can visually add a fourth dimension, "shade of gray," to the sheets to avoid their meeting in the fourth dimension. This has actually happened in the Neumann picture, where one sheet is white where they appear to meet and the other is dark gray.

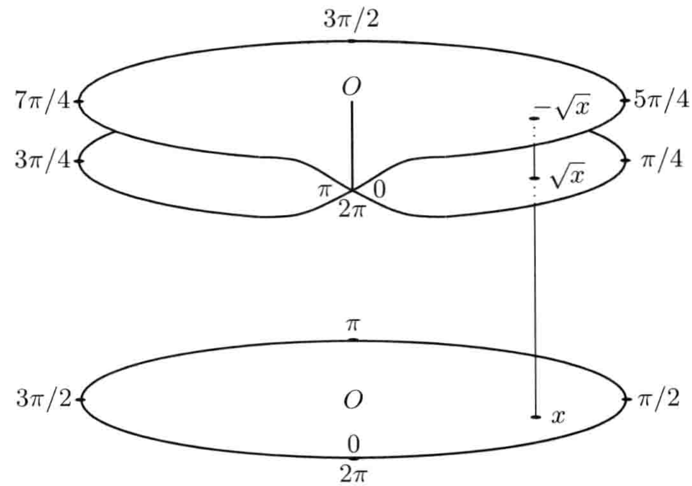

FIGURE 3. Branch point for the square root

Just as the curve $y^2 = x$ has a branch point of the two sheets at $x = 0$, the curve $y^n = x$ has a branch point of n sheets. An arbitrary algebraic curve $P(x, y) = 0$, where P is a polynomial of degree n, is an n-sheeted covering of $\mathbb{C}$ with a finite number of branch points. Since behavior of a curve at infinity is important, Riemann (1857) (Section 1, p. 117) extended $\mathbb{C}$ by a point ∞, and the resulting set $\mathbb{C} \cup \{\infty\}$ can be viewed as a *sphere* via the stereographic projection map shown in Figure 4. This idea is made explicit in Neumann (1865), p. 132. Under stereographic projection, each point $z \in \mathbb{C}$ corresponds to a point z' on the sphere other than the north pole N, and N itself naturally corresponds to ∞.

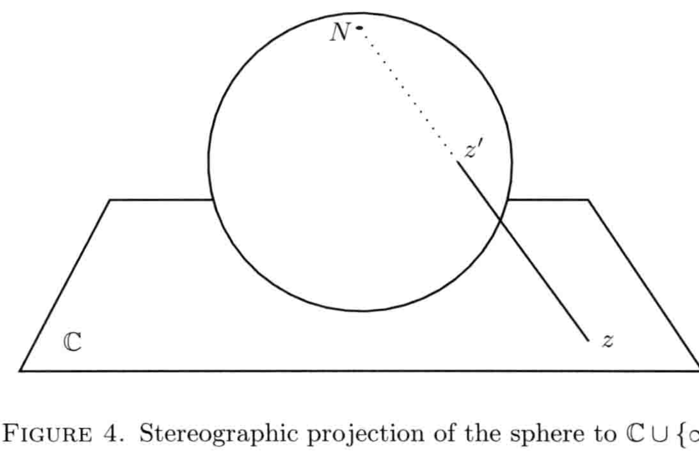

FIGURE 4. Stereographic projection of the sphere to $\mathbb{C} \cup \{\infty\}$

Corresponding to this completion of $\mathbb{C}$ to a sphere, we have a completion of each algebraic curve to a finite-sheeted covering of the sphere with finitely many branch points. Riemann realized that the covering surface S is topologically characterized by none other than Abel's number p, later dubbed the *genus* (or *Geschlecht* in German) by Clebsch (1865). Riemann described p topologically as half the number of closed cuts needed to make S simply connected (that is, such that any closed curve can be contracted to a point). In this case the resulting simply connected surface is a polygon with $4p$ sides. Möbius (1863) gave an even simpler interpretation of p, by showing that each Riemann surface is topologically equivalent to a member of the sequence of surfaces shown in Figure 5, namely, the one with p "holes."

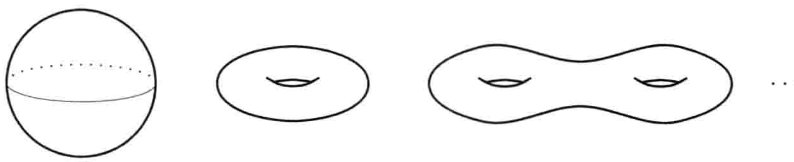

FIGURE 5. Riemann surfaces of genus 0, 1, 2, ...

As an example, consider the elliptic curve

$$y^2 = x(x-1)(x+1).$$

This curve is a two-sheeted cover of the sphere, with branch points like that shown in Figure 3 at $x = 0, 1, -1, \infty$. If we slit the sheets by cuts from 0 to ∞, and from 1 to -1, then, in order to obtain the branching, the edges of the cuts need to be identified so that the like-labeled edges shown in Figure 6 come together.

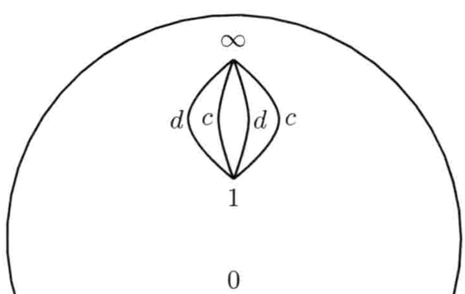

FIGURE 6. How edges are identified at branch points

But we can make a surface that is topologically the same by separating the two sheets before making the identifications, as shown in Figure 7.

The resulting surface is topologically a torus, shown in more familiar form in Figure 8. Thus Abel's number $p = 1$ agrees with the topological genus of the torus, because the torus has one "hole." (Notice that if 0 and ∞ are the only branch points, as is the case with $y^2 = x$, then the result of joining the two sheets is topologically a sphere, so the genus of $y^2 = x$ is zero.)

Moreover, as promised in the previous section, we can now see the reason for the two periods of elliptic functions associated with the curve $y^2 = x(x-1)(x+1)$. The periods are integrals over independent closed paths on the torus surface, such as the paths C_1 and C_2 shown in Figure 8.