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Lectures on Constructive Mathematical Analysis

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ЛЕКЦИИ ПО КОНСТРУКТИВНОМУ МАТЕМАТИЧЕСКОМУ АНАЛИЗУ

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Introduction(1)

§1.

As is well known, at the beginning of the twentieth century, thanks to the work of Cauchy, Bolzano, Weierstrass, Cantor, Dedekind, and Méray, mathematical analysis had obtained a foundation on the basis of Cantor's set theory. In our opinion, the two most significant features of the set-theoretic way of thinking are: 1) the admission of far reaching abstractions such as the abstraction of actual infinity, permitting us to consider "completed" infinite collections of simultaneously existing objects; and 2) the free application in arguments about infinite collections of the usual rules of traditional logic, in particular, the unrestricted application of the law of the excluded middle.

Set-theoretic methods permitted the passage from the vague "dynamic" concepts of the old infinitesimal analysis to the rigorous "static" system of concepts of the modern theory of limits. The developing, evolving natural number sequence was replaced by the idea of the collection of all natural numbers, and the process of becoming small, connected with infinitesimals, was reduced to the concept of function, which, in turn, is treated by means of an actually given "completed" set of pairs of objects, satisfying certain obvious restrictions (in the function set there must not be two different pairs with the same first component). The fact that the concepts are introduced in this way are really or seemingly natural and tangible, that they are convenient to deal with, due to the use of ordinary logical methods, stimulated to a considerable degree the development of mathematical analysis and created the impression

(1)This introduction should not be considered a sort of "constructivists' credo". Some of the opinions and evaluations expressed here reflect the personal point of view of the author, and are entirely his responsibility. A concise survey of the basic methodological ideas of the constructive approach in mathematics and a discussion of its position with respect to other approaches can be found in the papers of Markov [6], Shanin ([6], the Introduction and Appendix, and [8]), in the lecture of Tseĭtin, Zaslavskiĭ and Shanin [1]–[2] at the 1966 International Congress of Mathematics in Moscow, and, finally, in the author's summary [9] of Tseĭtin's dissertation.

of utmost rigor of its constructions, an impression amplified by the practical successes of the branches of applied mathematics based on analysis.

At the same time, the theory of sets, still in the process of construction, was shocked by paradoxes discovered on its outskirts (see, for example, Kleene [4], Curry [1], and Fraenkel and Bar-Hillel [1]). Although these paradoxes were not directly related to analysis (Richard's paradox (see Fraenkel and Bar-Hillel [1], Chapter I, §3.1) may be an exception because of its resemblance to Cantor's theorem on the uncountability of the continuum; an interesting discussion of this paradox can be found on p. 162 of Borel's book [1], nevertheless situations characteristic of the appearance of paradoxes already had been discovered in such an elementary part of analysis as the theory of real numbers. This and the extraordinarily great freedom of forming concepts (for example, Dedekind's continuum is the set of all sets of rational numbers satisfying certain rather weak restrictions), as well as the use of impredicative definitions, in which certain objects are defined in terms of sets to which they themselves must belong (for example, the definitions of least upper and greatest lower bounds of number sets are of this type).

On the other hand, independent of the problem of the paradoxes, there was an unceasing criticism, going back to Gauss and Kronecker, of the initial acceptability in principle of the basic set-theoretic ideas. Brouwer especially stood out for his sharp and consistent criticism. This criticism (later joined by Weyl who initially held a separate position) was accompanied by the development of an original program for the construction of mathematics, known now under the name "intuitionism" (or "neointuitionism"). Brouwer and his followers strongly objected to the faith in the existential character of infinite sets as well as to the belief that traditional logic corresponds to the essence of mathematics. According to the tenets of intuitionism, the objects of study in mathematics are mental constructions as such "without reference to questions regarding the nature of the constructed objects, such as whether these objects exist independently of our knowledge of them" (Heyting [3], p. 1).

Mathematical assertions are information about constructions that have been performed. Dealing with mental constructions requires a special logic, without accepting, in particular, the law of the excluded middle to its full extent (cf. Kolmogorov [2] and Heyting [3], pp. 1–2).

Intuitionism gave back to mathematical infinity its dynamic, developing character; the completed set of natural numbers, presented in toto for our consideration, ought to yield its place to the potentially infinite sequence of natural numbers, infinite in its development, in the possibility of constructing more and more new natural numbers. The continuum as a conglomerate of isolated points, which corresponds badly to geometric intuition, became a sort of "medium of formation", guaranteeing the possibility of unbounded development by means of acts of choice of freely forming a sequence of decreasing nested rational intervals. However, although intuitive clarity is, according to the position of the intuitionists, the principal and only criterion of mathematical truth, it is just this criterion which, in the opinion of many mathematicians, is often not satisfied by both the philosophical premises and the concrete mathematical theories of intuitionism. (For example, Bishop [2], [3] characterizes Brouwer's theory of the continuum as revolutionary and "semimystical").(2)

The polemics that developed around the paradoxes and the intuitionistic criticism revealed the serious divergence among the greatest mathematical thinkers in their views on the most fundamental and elementary concepts of mathematics, and created a situation that still exists, which can be characterized as a crisis in the foundations of mathematics (the problems involved here are presented in detail in the book of Fraenkel and Bar-Hillel [1], where there is also an extensive bibliography). Apparently it would not be an exaggeration to say that today it is not so clear whether the successes of applications are a consequence of the correct choice of the initial ideas of theoretical mathematics, or whether conversely these successes are themselves the source of the faith, shared by the overwhelming majority of mathematicians, in the correctness of these ideas. In light of what has been said it is natural to seek new ways of constructing analysis meeting the needs of the natural sciences and, at the same time, based upon clearer initial concepts than the set-theoretic ones.

Another side of intuitionistic criticism drew attention to the problem of constructivity in mathematics. The immense generality attainable in set theory led to poor "tangibility" of many objects of analysis. Moreover, such objects arose not only as a result of risky constructions using the axiom of choice, but also appeared in the most elementary parts of analysis directly related to computational practice. Many such "existence theorems" turned out, upon closer examination, to be devoid of computational meaning. As a classical example, let us turn to the theorem on least upper and greatest lower bounds of bounded number sets. It is not difficult to construct an algorithmic sequence $\{n_k\}$ of zeros and ones such that a nonzero term occurs in this sequence when and only when Fermat's last theorem is violated. (For this it suffices to enumerate one after the other all quadruples of natural numbers (x, y, z, n) $(n > 2,\ x, y, z > 0)$ and check the equation $x^n + y^n = z^n$; n_k is set

(2)For a detailed acquaintance with the philosophy and mathematical practice of intuitionism, one can turn to Heyting's book [3], already cited, or to the monograph of Fraenkel and Bar-Hillel [1].

equal to 0 if the result of this check for the kth quadruple is negative, and to 1 otherwise.) According to the Bolzano-Weierstrass theorem, there exists a least upper bound b of the set of values of the sequence $\{n_k\}$. It is clear that being able to compute b even with only an accuracy of $\frac{1}{3}$ we would be able to find whether b is equal to 0 or 1. Thus, we would find out whether Fermat's last theorem is true or not. Similar examples—and it is easy to construct many more—make it very doubtful that there is an effective method enabling us to compute the least upper and greatest lower bounds of bounded sets, even if we consider only sets of values of effectively computable sequences of zeros and ones.(3)

As a second example let us take the theorem on a zero of an alternating continuous function. (An *alternating* function on an interval is one that takes values of opposite signs at the endpoints of the interval.) Here it may seem that the usual method used in this proof, successive division of the interval (see, for example, Fikhtengol'ts [1], Chapter 2, §5), enables us to effectively find arbitrarily precise approximations to a root. In fact, the matter is not so simple: at the very first step in the computation of the zero one must find whether the function vanishes at the center of the interval. However, it is easy to see that the problem of determining whether a real number is equal to 0 can be immensely difficult. In fact, let us denote by a the real number whose integral part is equal to 0 and whose kth binary digit is equal to n_k (where $\{n_k\}$ is the sequence of zeros and ones considered above). It is clear that $a = 0$ if and only if Fermat's last theorem is true. The difficulties in finding a zero of an alternating function that have been revealed by this argument are connected, as will be shown in §4 of Chapter 5, not with special properties of the method of successive subdivision of an interval,(4) but stem from the core of the matter.

In both cases that we have considered, it is not hard to find in the proofs of

(3)The example that we have considered is convenient for elucidating the intuitionistic denial of the law of the excluded middle. Since Fermat's last theoem has so far been neither proved nor disproved it is impossible to assert constructively (constructive mathematical assertions are assertions about performed constructions') that the sequence $\{n_k\}$ has a least upper bound. The reason serving as a basis of the traditional proof, that either $n_k = 0$ for all k or $n_k \neq 0$ for some k, must be rejected as metaphysical, appealing to truth in some absolute sense, and, therefore, going beyond the boundaries of mathematics. Of course, in time Fermat's last theorem may be either proved or refuted, but then one can simply take another unsolved problem. The confidence in our ability to eventually solve any problem is possibly an interesting theme for philosophical discussions, but, in any case, it should not be a source of mathematical theories.

(4)With minor changes, this method can be made the basis for an algorithm that computes the zeros of alternating continuous computable functions satisfying certain additional conditions.

the corresponding classical theorems an application of the following variant of the law of the excluded middle: either all elements of a set $\mathcal{A}$ possess a given property $\mathcal{C}$, or there exists some element of $\mathcal{A}$ not possessing this property. Bishop [2] picturesquely calls this principle the "principle of omniscience" and considers it the chief culprit of nonconstructivity in classical mathematics.

The examples we have presented induce a natural desire to refine a number of the concepts that have to do with computability. For example, which real numbers and functions are to be considered computable? What are their properties? What is to be meant by a "general effective method" for computing the least upper and greatest lower bounds of bounded sets (or for computing zeros of alternating functions)? What should the initially given data be that are used by such a method, and how are they to be presented, and is it possible to somehow refine the intuitively felt impossibility of such a method? Anticipating somewhat, we mention that, after the concepts in the given examples have been refined in a proper way, one obtains the following situations in the above examples. In the case of the theorem on least upper and greatest lower bounds, one can produce an example (Specker [1]) of a bounded, increasing algorithmic sequence of rational numbers not having a computable least upper bound. (Thus, not only a general method for computing least upper bounds is impossible, but there occur particular, quite simple sets not having computable least upper bounds.) In the case of Cauchy's theorem, an algorithm which finds the zeros of any alternating, continuous, computable function is impossible. (As initial data such an algorithm should use an algorithm computing the given function. We note that the desired algorithm is impossible even for the class of piecewise linear functions of the form $f(x) + c$, where c is a computable real number and f is the function whose graph is presented in Figure 1.) At the same time, there cannot be a computable function that assumes values of different signs at the ends of a given interval and does not vanish at any computable point of this interval (a priori it is impossible to rule out the existence of computable alternating functions, whose zeros are all "noncomputable"). These results are due to Tseĭtin [2], [6].

Briefly summarizing what we have said, we can single out the following two circles of problems.

(1) The construction of a system of analysis based on premises clearer than those of set theory and taking into account, to a greater degree than traditional analysis, the actual constructive and computational possibilities.

(2) The introduction and study of computable objects of analysis, the investigation of the theoretical limitations of computational possibilities in analysis, the study of "effectivity" in analysis, and, in particular, the investigation of

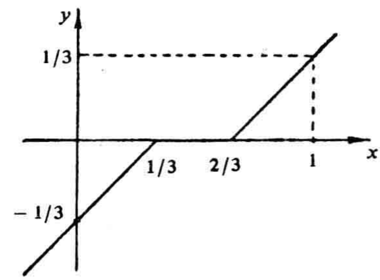

FIGURE 1

the question as to which initially given data suffice to find various objects of analysis.

In connection with these problems there arose various currents in the foundations of mathematics and mathematical analysis, joined under the collective name "constructive analysis". (One uses the terms "recursive analysis" and "computable analysis" in a similar sense.)(5) In addition, while the investigations of problems (1) are connected with the development of various concepts in the foundations of mathematics, the investigations of problems (2) can also be carried out by the usual set-theoretic means.

§2.

Proceeding to a brief historical review, let us mention immediately the immense service performed by intuitionism in the formation of the main concepts of constructive analysis. Also of great importance was Weyl's book [1], already mentioned; this book contained, in particular, one of the first approaches to the concept of a constructive real number.

A very substantial step in the development of constructive analysis was the elaboration in the 1930s of the precise concept of algorithm, thanks to the work of Herbrand, Gödel, Turing, Post, Church, and Kleene. (In fact, several outwardly different concepts were proposed; however, they turned out to be equivoluminous.) Basing themselves on precise concepts of algorithm (Turing machines and recursive functions, respectively), Turing [1]–[2] and Banach

(5) We do not here go into the subject of "predicative analysis", which had it beginning in the monographs of Whitehead and Russell [2]–[3] and Weyl [1]. In predicative analysis, the objects dealt with are introduced by way of individual definitions within a framework of certain fixed means; in this connection, special attention is paid to the exclusion of impredicative formation of concepts (for more details, see Feferman [1]).

and Mazur [1](6) independently offered in 1936–37 definitions of a computable (constructive) real number. In his first paper, Turing defined a computable real number as a number admitting a computable decimal expansion.(7) In a subsequent correction (Turing [2]), this definition was changed; it was found that the (at first glance) natural linking of computability of numbers with computability of their systematic representations in a fixed number system had a host of fundamental deficiencies. We shall indicate only two of them: 1) for any two number systems there may not be an algorithm which enables us to go over from computable expansions in the first system to computable expansions in the second (see the Mostowski-Uspenskiĭ theorem in §3 of Chapter 4); 2) for any fixed number system, there is no algorithm for adding real numbers which are computable with respect to this system. Dispensing with technical details, one can say that, in his revised definition, Turing connects the computability of a real number x with the existence of a computable sequence $\mathfrak{A}$ of rational numbers (that is, an algorithm transforming every natural number into a rational number)(8) such that, for every n,

$$|\mathfrak{A}(n) - x| \leq 2^{-n}. \tag{3}$$

Although this definition cannot be accepted under a thoroughly constructive interpretation because of the traditional concept of real number occurring in it, the idea contained in it easily enables us to define constructive real numbers from the beginning without appealing to other concepts of real numbers. To do this it suffices to replace (3) by the condition

$$|\mathfrak{A}(n) - \mathfrak{A}(m)| \leq 2^{-n} \quad \text{for } m \geq n, \tag{4}$$

and to mean by constructive (computable) real numbers computable sequences of rational numbers satisfying (4).

The concept of constructive (computable) real number just presented (which the constructive numbers considered in this book also satisfy) apparently can be considered to be definitive. It is interesting to notice that Turing mentions the influence on his definition of some ideas of Brouwer.

The investigations begun by Turing, Banach and Mazur were continued in the postwar years. In 1949 there appeared a paper by Specker [1] in which,

(6)Reference [1] gives only the title of a lecture given in Lwow on January 23, 1937 at a meeting of the Polish Mathematical Society. Some information about its content can be found in Mostowski's survey [1].

(7)According to Mostowski [1], Banach and Mazur considered real numbers with primitive recursive decimal expansions.

(8)Of course, we have in mind here an algorithm in the precise sense of the word (for example, a Turing machine). We remark that computability of sequences of rational numbers reduces in an obvious way to computability of arithmetic functions (that is, functions with natural numbers as arguments and values).

along with a profound study of primitive recursive computable objects of analysis (real numbers and functions), there was given a famous example of a nondecreasing bounded algorithic sequence of rational numbers which is not computably convergent. (This result has already been mentioned above in connection with the problem of least upper bounds.) In more precise terms, for Specker's sequence $\mathfrak{S}$ there cannot exist a general recursive function h such that, for any i, j and n satisfying the inequality $i, j \geq h(n)$,

$$|\mathfrak{S}(i) - \mathfrak{S}(j)| < 2^{-n}.$$

Thus, the rate of convergence of this sequence (meaning the convergence asserted by the well-known classical theorem) cannot be effectively estimated.

Various representations of computable real numbers were studied further in the papers of Péter [1] (1950) (Péter's results also were presented in her well-known monograph [2]), Myhill [1] (1953), Meschkowski [1] (1956), and Rice [1] (1954). Rice's paper presents, in particular, an especially lucid construction of the Specker example considered above.

In the academic year 1949-1950, in a lecture course at the Institute of Mathematics of the Polish Academy of Science, Mazur gave a thorough presentation of the results on computable analysis obtained by him before the war jointly with Banach, as well as his own postwar results. The notes of these lectures were published later (1963), with the help of Rasiowa and Grzegorczyk, in the form of a monograph (Mazur [1]). Mazur's monograph contains, in particular, a concise and profound presentation of the theory of computable real numbers and functions. Along with the general concept of computable real number, similar to Turing's and parallel to Cantor's definition in traditional analysis, this monograph also studies other possibilities of defining a computable real number (systematic expansions, Dedekind cuts). Primitive recursive computability is also studied.

Mazur identifies intuitive computability in the domain of natural numbers with general recursiveness. His approach to the definition of computable objects of more complex types is very distinctive. For example, let us take a look at the definition of computable functionals over arithmetic functions with natural numbers as values. A functional (in the traditional sense—Mazur freely uses concepts of set-theoretic mathematics) is said to be *computable* if it transforms every computable sequence of computable arithmetic functions (such a sequence is given by a two-place general recursive function) into a computable sequence of natural numbers. In more precise terms, for a comutable functional Φ, for any two-place general recursive function f there exists (the proof of this existence can be carried out by using any mathematical means) a general recursive function g such that $g(n) = \Phi(f(n, m))$.

(Here, $f(n, m)$ is considered as a function of m for every fixed n.) In this order of ideas, one also defines computability of real functions: a function φ is computable if it transforms every computable sequence of computable real numbers into a computable sequence of computable real numbers. Here, computable sequences of computable real numbers are interpreted by means of two-place computable (jointly in both arguments!) approximating rational functions. (This is actually equivalent to our definition of sequences of constructive real numbers in §1 of Chapter 3.) One of the remarkable results presented in Mazur's monograph is a theorem on the continuity of computable (in the sense indicated above) functionals and real functions. This theorem is close to a theorem of Markov on the impossibility of constructive discontinuities for computable real functions (for a somewhat different concept of computable function; see Markov [3], [5]). Mazur's study served as a starting point for a series of papers (Grzegorczyk [2]–[5], Mostowski [2], Lehman [1], Lachlan [1], Friedberg [2], Klaua [1]–[4] and Ilse [1]), whose extent, depth, and distructiveness permit us to speak about a Polish school of computable analysis.

In Mostowski's paper [2] (1957), various ways of defining computable sequences of computable real numbers were studied; among the results obtained is a theorem completely solving the problem of the possibility of an effective passage from computable expansions in one number system to computable expansions in another system. Such a passage from a system with base m to a system with base n turns out to be possible if and only if all prime divisors of n are prime divisors of m (see §3 of Chapter 4; this theorem also was found independently by Uspenskiĭ [2], [3]). Some questions left open in this paper were solved by Lehman [1] and Lachlan [1].

In the fundamental paper of Grzegorczyk [2] (1955), a new and original approach to the definition of computable real function was proposed. The starting point is everywhere defined computable functionals over natural numbers and arithmetic functions, with natural numbers as values. The Grzegorczyk-computable functionals have a genetic character: every such functional is obtained from very simple initial functionals by means of a finite number of applications of certain rules (similar to the corresponding rules in the theory of partial recursive functions.)(9) Starting from functionals with natural numbers as values, one can easily introduce in a natural way computable functionals over integral-valued functions, with integers as values.

In Grzegorczyk's definition the computability of an everywhere-defined real function φ is connected with the existence of a Grzegorczyk-computable func-

(9) As observed by Kellene [3], a Grzegorczyk functional is identical with an everywhere-defined general recursive functional (see Kleene [4]).

tional transforming every integral-valued function which approximates an arbitrary real x into a function approximating $\varphi(x)$. (A function f (not necessarily computable!) approximates x if, for all n,

$$\left| x - \frac{f(n)}{(n+1)} \right| < \frac{1}{(n+1)}.$$

From the definition of Grzegorczyk functionals, their continuity (in the Baire metric) follows: for fixed natural number arguments, the value of a computable functional at given functions $f_1, \ldots, f_n$ is determined by some initial interval of values of these functions. Using this fact and the fact that computable functionals are everywhere defined Grzegorczyk proved with the help of a theorem of Borel on coverings that a real function which is computable in his sense is computably uniformly continuous on every segment. Grzegorczyk's ideas were developed further by the German mathematician Klaua; Klaua's results are summarized in his monograph [4].

Beginning in 1955, there were published, primarily in the Comptes Rendus of the French Academy of Sciences, a series of papers by Lacombe, Kreisel, Shoenfield, and Friedberg (also see the Proceedings of the Colloquium on Constructivity in Mathematics (Amsterdam, 1957; Heyting [4])). Among the fundamental results obtained by these authors, let us mention the following.([10]) Kreisel, Lacombe and Shoenfield [1], [2] proved the computable continuity of effective functionals over the Baire space of general recursive functins. Lacombe [2] established the existence of computable real functions which are not uniformly continuous. Lacombe [2], [4] constructed an example of a computably uniformly continuous, computable function which does not achieve its maximum at any computable point (the example constructed by Kleene [2] of an infinite tree with finite branching, all of whose general recursive paths are finite) may serve as the source. Kreisel and Lacombe [1] proved the existence of singular interval coverings (see §1 of Chapter 8).

The characteristic feature of these papers is the fact that their athors were mainly interested in the circle of problems (2) of §1, and do not find themselves by any specific concepts in the foundations of mathematics. Concepts, methods, and results of set-theoretic mathematics are freely used in definitions and proofs. The advantage of this approach is its great flexibility, which enables the authors in question, to obtain, in particular, very interesting results characterizing the relationships between computable and noncomputable objects, as well as the possibility of presenting computable analysis on the basis

([10])Similar results were obtained at the same time in the Soviet Union by I. D. Zaslavskiĭ and G. S. Tseĭtin (Zaslavskiĭ [1], [2], [4] on computable functions with unusual properties, Tseĭtin [3], [5] on continuity theorems, and Zaslavskiĭ and Tseĭtin [1], [2] on singular coverings). The Soviet school of constructive analysis will be discussed below.

of ordinary language and the well-developed symbolism of traditional mathematics. At the same time, the criticism to which the set-theoretic way of thinking has been subjected also carries over to these results, which makes this method poorly suited for researchers interested in the circle of problems (1) (that is, the initial "self-contained" construction of constructive analysis). We must mention that, in many cases, set-theoretic methods can be eliminated, which permits us to use a significant part of the achievements of the traditional systems of computable analysis within a thoroughly constructive approach. However, the matter is not always so simple: for example, the definition of Grzegorczyk-computable real functions is so permeated by set-theoretic concepts (arithmetic function, real number) that even the formulation of meaningful analogues of this definition, not to speak of the theorem on uniform continuity, within the framework of a nontraditional system of analysis (like the one developed in this book, say), appear nontrivial.

Along with the papers that we have considered, attempts were also made to construct nontraditional systems of computable analysis. It is necessary here to talk first of all about Goodstein's recursive analysis.(11) (Goodstein's first paper on this subject, submitted in 1941, actually appeared in 1945; Goodstein's investigations were collected in two of his monographs [1] and [2], combined into one book in the Russian translation [5].) The characteristic feature of Goodstein's approach is the tendency to construct recursive analysis on the simplest possible logical basis—namely, on the basis of primitive recursive arithmetic or, more precisely, on the basis of an original axiomatic theory developed by him for certain classes of arithmetic functions (Goodstein's equation calculus; for a more detailed general description of the equation calculus and of Goodstein's approach, we refer the reader to a paper of Shanin [8]). The objects under consideration are recursive (most often, primitive recursive) functions over the field of rational numbers with rational values, as well as recursive sequences of such functions, given by two-place recursive functions. In the construction of analysis, Goodstein consistently employs approximation methods, whereby objects which usually arise in analysis from approximations as a result of limit operations are not, as a rule, introduced. For example, the concept of recursive real function actually is absent from Goodstein's work, although a reader possessing some conception of real function can without difficulty find sequences of rational approximations which lead to such functions. The merit of this methodology is the logical simplicity of the concepts that are used; at the same time, it is not devoid of certain deficiencies, in the opin-

(11)We shall not deal now with intuitionistic analysis, which occupies quite a special place (see Heyting [3]); its fruitful influence on practically all researchers in this area has already been mentioned.

ion of the author—as the number of limit passages that are not designated in any way increases, so does the awkwardness (but not the complexity!) of definitions and formulations of theorems. On the whole, analysis acquires a very unusual form that makes it substantially more difficult for a mathematician interested in the circle of problems (2) from §1 to relate the results thus obtained to familiar mathematical structures.

It should be noted that Goodstein's papers apparently were the first to study mean-value theorems systematically from the point of view of a rigorous theory of algorithms, and to suggest a fruitful approach to establishing recursive analogues of these theorems, which, instead of an object giving the desired value of the function, determined a recursive object which gives the required value to within a preassigned fixed accuracy. This kind of ε-variant of existence theorems is sometimes called a theorem of Goodstein type.

In 1966–67 the well-known American mathematician Errett Bishop came forward with an original and extremely advanced system of constructive analysis (see his monograph [2] (1967) as well as the summaries [1] and [3] of his lecture at the 1966 International Congress of Mathematicians in Moscow). Bishop's constructive analysis occupies an intermediate position between intuitionistic analysis and systems using a precise concepts of algorithm. Allying himself with the intuitionistic criticism of set-theoretic mathematics, Bishop, at the same time, tends to avoid what he calls "preoccupation with the philosophical aspects of constructivism at the expense of concrete mathematical activity". He rejects intuitionistic theorems like Brouwer's fan theorem, which implies the uniform continuity of intuitionistic real functions, as well as claims that the precise notions of algorithm give complete expression to the essence of computability (Church's thesis, Turing's thesis, the principle of normalization). All of them contain supramathematical, and therefore unacceptable, assumptions. Bishop develops his constructive analysis, up to the presentation of deep results concerning the theory of functions of a complex variable and functional analysis, on the basis of an intuitive concept of constructivity, assuming, in particular, initial intuition of the natural numbers and their arithmetic, and the view that every mathematical assertion must, in the final analysis, express some fact of a computational nature about the natural numbers (this or that computation on natural numbers yields this or that result). Bishop's monograph [2], written with great pedagogical skill, permits us to assert the definite and significant success of this program. This fact, however, in no way diminishes the importance of investigations using rigorous concepts of algorithm. Those investigations have the advantage of greater precision in the formulation of problems and the possibility of proving the undecidability of many natural algorithmic problems,(12) and, finally, the possibility of studying specific properties of objects that are computable in the precise sense. That the results obtained here are interesting is obvious, since even the few mathematicians who, like Bishop, deny the absolutistic claims of the rigorous concepts of algorithms apparently recognize their very great generality.

It appears very plausible that most of Bishops's results can also be interpreted within systems of analysis based on a precise concept of algorithm.

The system of constructive analysis presented in this book belongs to the so-called constructive approach to mathematics, the principal tenets of which were suggested in 1948–1949 by A. A. Markov. The formation of the basic concepts of this system took place in the 1950s; at that time were also obtained the most fundamental results, which determined the contemporary features of the theory. Here it is necessary to indicate, first of all, the ground-breaking contribution of A. A. Markov himself, as well as those of his students N. A. Shanin, I. D. Zaslavskiĭ, and G. S. Tseĭtin. A brief characterization of the constructive approach and of the constructive analysis built up within its framework will be presented in the following sections. Below, the adjective "constructive" will be used (except in specially designated cases) to indicate that this or that method, result, etc. belongs to the constructive approach to mathematics. In particular, the term "constructive analysis" is assigned to the system presented by us.

Finishing our brief survey, we note that even very remote areas of mathematics have been studied from the points of view that we have mentioned (especially in recent years). In this connection, in addition to Bishop's monograph and the chapters of Martin-Löf's monograph [1](13) devoted to measure theory, one can mention the approach to the formulation of a constructive theory of Lebesgue measure and integral proposed by Shanin [2], [6], the series of papers devoted to the same theme by the Czechoslovak mathematician Demuth [1]–[16], the papers of Kosovskiĭ [1]–[4] and Lorents [1] on the constructive theory of probability, the papers of Manukyan [1] and Lifshits [4], [5] on constructive functions of a complex variable, the papers of Orevkov [1], [2], [4], and Zaslavskiĭ and Manukyan [1] on combinatorial topology, and the work of Phan Đình Diệu [1]–[9] on constructive linear topological spaces and generalized functions (the papers just enumerated belong to the constructive

(12)Here there is a definite similarity with the progress achieved by use of the rigorous concepts of algorithm in the study of well-known algorithmic problems of logic, number theory, algebra, and topology.

(13)This immensely interesting monograph, which also contains a concise outline of the elementary problems of computable analysis, was unfortunately not available to the authors while he was working on this book.

approach), and also the investigations of Lacombe [6] and Nogina [1]–[3] on recursive general topology.

Finally, thanks to the work of A. N. Kolmogorov and his students (Martin-Löf, Levin, and others; see the survey article by Zvonkin and Levin [1]), it has recently been made clear that a fruitful application of the theory of recursive functions to the foundations of information theory and probability theory is possible.

§3.

The constructive approach to mathematics can be characterized by the following main features (cf. Markov [6], and Tseĭtin, Zaslavskiĭ and Shanin [1], [2]).

I. The objects of study are constructive objects, in dealing with which the abstraction of potential realizability is admissible but the abstraction of actual infinity is completely excluded.

II. The intuitive concepts of "effectivity", "computability", etc. are connected with a precise concept of algorithm.

III. One uses a special interpretation of mathematical judgments which takes into account the specific nature of constructive objects.

The constructive approach arose from roughly the same basis of criticism as intuitionism; at the same time, the positive programs of these two movements have fundamental differences. Constructivism is very remote from the philosophical premises of intuitionism as well as from specific intuitionistic theories—especially those theories which use free-choice sequences and which play one of the central roles in intuitionistic mathematics. The widespread belief in the closeness (or even the identity) of intuitionistic and constructive mathematical practice must be recognized as a profound error.

On the whole, constructive mathematics uses a considerably more "modest" system of abstractions than traditional mathematics. This fact, however, is not in itself evidence of limited possibilities for applying constructive theories. The external world does not impose on us the necessity of substances more general than constructive objects, or the idea of actual infinity (the latter has already been pointed out by Hilbert [1]). In fact, the constructive analysis developed within the constructive approach can already serve as a theoretical basis for the usual applications of differential and integral calculus. The development of events in this direction has been held back both by educational traditions and by the comparative awkwardness of constructive theories. Whether this awkwardness is an inevitable "payment for construc-