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Part 3
Structural Analysis, Fatigue

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STRUCTURAL ANALYSIS

A SOURCE OF LARGE ERRORS IN CALCULATING SYSTEM FREQUENCIES

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Attention is called to errors in system frequency calculations resulting from the use of non-diagonal mass matrices with eigenvalue routines that replace the mass matrix with its eigenvalues before proceeding to the calculation of system frequencies. The errors are illustrated in several different solutions of an 18 degree-of-freedom system. What can be done to avoid these errors is presented.

In some specific instances recently, the author has observed errors in the determination of system frequencies by factors of 3 to 10. These errors were costly in replacement or retrofit, so an investigation was undertaken to determine the cause of the errors. In each case, the Archer formulation of consistent mass was computer generated, and so was the stiffness matrix. These matrices were then run through one of several widely used eigenvalue routines to get system frequencies and mode shapes. If the stiffness and mass matrices had been repetitive, all would have been well, but in these cases both the stiffness and mass matrices had considerable texture (more than 1000 to 1 variation in magnitudes). Further examination of the eigenvalue routines revealed that they used the device of formulating $M^{-\frac{1}{2}} * K * M^{-\frac{1}{2}}$ to achieve symmetry. For a diagonal mass matrix, $M^{-\frac{1}{2}}$ is no problem, but with the non-diagonal mass matrices generated in the programs, the eigenvalue routines first found the eigenvalues of the mass matrix and used the inverse square root of these eigenvalues to get a diagonal $M^{-\frac{1}{2}}$. Unfortunately, the routines ordered the eigenvalues of the mass matrix and scrambled the coordinate order in the process. Since the eigenvalues of the mass matrix constitute a transformed set of coordinates, they cannot be used to multiply the untransformed stiffness coordinates.

To study this problem further, the system shown in Fig. 1 was used. This system was chosen for two reasons: first, the author had used this system for several years as a check problem on different computers with consistent results; second, the stiffness had a range of 2250 to 1, and the mass had a range of 5400 to 1. If the foregoing hypothesis were correct, this system should show it strongly. The main diagonals of stiffness, diagonal mass, and consistent (Archer) mass are shown in Table 1, together with the eigenvalues of the consistent mass matrix.

Eight different solutions for eigenvalues were made with the results shown in Table 2, in which

Column 1. Frequencies from $M_D^{-\frac{1}{2}} * K * M_D^{-\frac{1}{2}}$, with EIGN5/RMM, M_D = diagonal mass matrix by eyeball.

2. Frequencies from $K * M_D^{-1}$, with EIGRF/IMSL, wholly real.

3. Frequencies from $K * M_C^{-1}$, with EIGRF/IMSL, M_C = consistent mass per Archer, wholly real.

4. Frequencies from $M_E^{-\frac{1}{2}} * K * M_E^{-\frac{1}{2}}$, with EIGN5/RMM, M_E = eigenvalues of M_C arranged in near natural order by eyeball.

5. Frequencies from $M_{DA}^{-\frac{1}{2}} * K * M_{DA}^{-\frac{1}{2}}$, with EIGN5/RMM, M_{DA} = diagonal mass arranged in ascending order.

6. Frequencies from $M_{EA}^{-\frac{1}{2}} * K * M_{EA}^{-\frac{1}{2}}$, with EIGN5/RMM, M_{EA} = eigenvalues of M_E arranged in ascending order.

7. Frequencies from $M_{DD}^{-\frac{1}{2}} * K * M_{DD}^{-\frac{1}{2}}$, with EIGN5/RMM, M_{DD} = diagonal mass arranged in descending order.

8. Frequencies from $M_{ED}^{-\frac{1}{2}} * K * M_{ED}^{-\frac{1}{2}}$, with EIGN5/RMM, M_{ED} = eigenvalues of M_E arranged in descending order.

EIGRF/IMSL "calculates eigenvalues and eigenvectors of a real, general matrix," not

required to be symmetric. Eigenvalues and eigenvectors may be complex.

EIGN5/RMM calculates eigenvalues and eigenvectors from $M^{-\frac{1}{2}} * K * M^{-\frac{1}{2}}$. When M is not diagonal, the eigenvalues of M are used instead and are ordered ascending or descending.

NROOT, from the IBM system 360 Scientific Subroutine Package (SSRP), "finds the eigenvalues and eigenvectors of a real, square non-symmetric matrix of the special form $B^{-1} * A$, where both B and A are real, symmetric matrices and B is positive definite." This routine calls EIGEN from the SSRP to determine the eigenvalues of B, orders them in descending order, and then finds the eigenvalues of $B^{-\frac{1}{2}} * A * B^{-\frac{1}{2}}$.

In Table 2, the first two columns were identical all the way. Columns 3 and 4 were close to 1 and 2 for the lower third of the frequencies and differed appreciably thereafter. Columns 5 and 6 are close to each other in the lower third, but a factor of 10 different from 1 and 2. The same comments apply to columns 7 and 8 as to 5 and 6.

The eigenvector sets for the various solutions were too voluminous to be reproduced here except for eigenvectors number one and two from solutions 1 and 2, which are shown in Table 3. While the frequencies from 1 and 2 were identical, the vector shapes were radically different, so that further calculations involving the vector shapes (as for stress) would also be different.

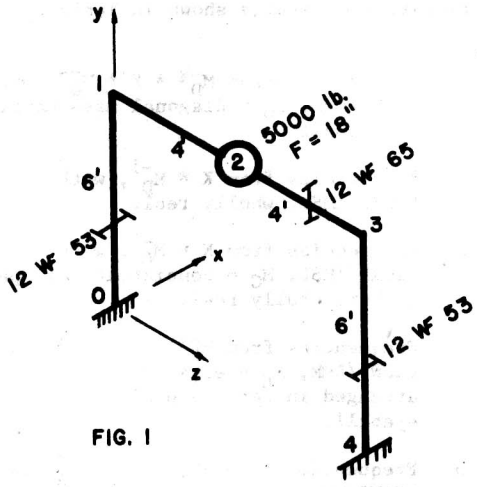

FIG. 1

From the numbers in Table 2, it seems clear that the use of a non-diagonal mass matrix in dynamic system analysis can lead to large errors if the eigenvalue routine uses the eigenvalues of the mass matrix in place of the matrix. If the $K * M^{-1}$ formulation is used with a routine like EIGRF/IMSL, the eigenvalues are correct, but the eigenvectors are not orthogonal and subsequent calculations need to be modified. Both of these problems can be gotten around if the non-diagonal mass matrix is replaced by its triangular decomposition such that $M = L * L^T$. This replacement permits the eigenvalue solution of $L^{-1} * K * L^{-T}$, which is symmetric and has orthogonal eigenvectors, so that subsequent calculations can be carried through the same as for a diagonal mass matrix. The use of the triangular decomposition of the mass matrix was called to the author's attention by Mr. Eugene Sevin of the Defense Nuclear Agency, and the author gratefully acknowledges this.

TABLE 1

Diagonals of Matrices Used

Diagonal of Stiffness lb/in	Diagonal Mass lb sec^2/in	Diagonal of Consistent Mass lb sec^2/in	Eigenvalues of Consistent Mass lb sec^2/in
0.13491+07	0.77700+00	0.57720+00	0.12211+00
0.98719+09	0.77700+01	0.77700+01	0.77447+01
0.76931+07	0.77700+00	0.57720+00	0.12211+00
0.50572+09	0.77700+01	0.77700+01	0.77447+01
0.16010+08	0.77700+00	0.57720+00	0.12211+00
0.52064+09	0.77700+01	0.77700+01	0.77447+01
0.21801+07	0.12950+02	0.12950+02	0.13430+02
0.30418+10	0.42000+04	0.42000+04	0.42000+04
0.23992+07	0.12950+02	0.12950+02	0.13430+02
0.22293+10	0.42000+04	0.42000+04	0.42000+04
0.31847+08	0.12950+02	0.12950+02	0.13430+02
0.12500+08	0.42000+04	0.42000+04	0.42000+04
0.13491+07	0.77700+00	0.57720+00	0.12277+00
0.98719+09	0.77700+01	0.77700+01	0.82076+01
0.76931+07	0.77700+00	0.57720+00	0.12277+00
0.50572+09	0.77700+01	0.77700+01	0.82076+01
0.16010+08	0.77700+00	0.57720+00	0.12277+00
0.52064+09	0.77700+01	0.77700+01	0.82076+01

TABLE 2

Frequencies for Different Solutions, Hz

1	2	3	4	5.	6	7	8
8.50	8.50	8.50	8.51	0.95	0.95	0.75	0.80
16.12	16.12	15.95	16.31	11.59	11.44	6.25	6.25
16.22	16.22	16.00	16.70	21.60	22.36	13.34	13.40
35.46	35.46	37.27	42.61	51.97	55.98	25.24	26.30
50.46	50.46	50.11	49.57	59.39	62.37	54.41	53.67
117.9	117.9	124.2	127.1	64.37	64.45	94.25	93.01
126.7	126.7	126.5	274.1	115.6	115.2	97.08	199.3
142.4	142.4	140.0	283.9	117.2	195.1	200.7	228.5
484.1	484.1	584.3	1186.	201.4	220.8	279.0	285.5
485.6	485.6	587.5	1192.	217.1	248.8	473.8	855.8
722.2	722.2	723.9	1264.	473.6	1188.	760.7	991.1
763.9	763.9	784.5	1301.	760.5	1356.	864.6	1186.
1293.	1293.	3039.	1334.	1378.	1830.	1009.	1836.
1294.	1294.	3049.	1369.	2317.	2371.	2367.	2486.
1306.7	1306.7	3221.	1786.	3215.	3149.	3119.	3054.
1306.8	1306.8	3226.	1814.	4121.	10285.	4120.	10317.
1798.9	1798.9	3668.	1837.	4292.	10366.	4293.	10394.
1799.2	1799.2	3705.	1861.	5692.	14319.	5692.	14280.

TABLE 3

Comparison of Eigenvectors

Eigenvector no.1		Eigenvector no.2	
EIGRF	EIGN5	EIGRF	EIGN5
+0.47581-02	-0.24729-01	+0.55669-01	+0.21024+00
-0.13071-15	-0.60205-10	-0.38610-13	-0.23599-08
+0.34562-15	-0.74144-09	-0.94688-14	-0.60161-08
+0.11659-03	-0.27964-02	+0.58575-02	+0.25372-01
-0.20680-14	-0.32534-07	-0.11484-11	-0.70346-06
-0.12083-02	-0.66189-03	-0.99394-02	+0.71654-02
+0.82262-01	-0.26185-01	+0.11770+01	+0.26594+00
+0.88150-14	-0.16893-10	+0.17071-11	+0.17459-09
-0.45832-15	-0.23894-08	-0.20972-14	-0.91465-09
-0.24881-07	-0.83066-05	-0.36160-06	+0.70316-04
-0.36048-13	-0.32728-07	-0.19227-10	-0.70646-06
-0.15969+02	+0.15354-01	+0.21944+01	+0.15332-02
+0.47581-02	-0.24729-01	+0.55669-01	+0.21024+00
+0.12481-15	-0.15504-09	-0.38603-13	-0.23962-08
+0.27998-15	+0.42277-10	+0.91848-14	+0.57532-08
-0.11659-03	+0.60594-04	-0.58575-02	-0.22122-02
-0.22953-14	-0.32657-07	-0.11491-11	-0.70350-06
-0.12083-02	+0.62796-03	-0.99394-02	-0.37538-02

DISCUSSION

Mr. Gupta, (IIT Research Institute): I was curious as to which programs the two consultants used. I am really skeptical in the sense that there are many commercial programs which are available and usually they do try to solve problems and show that they are able to solve certain problems; but having a error of the order of magnitude which you showed certainly is not the answer.

Dr. Mains: In the case of the floor system the programs that were used and the people doing the analysis are a part of one of the largest dynamic analysis operations in the country. They purport to do this kind of thing all day every day and I know that they have an automatic consistent mass calculator built into their programs. Their programs also substitute the eigenvalues of the mass matrix in that particular case. The other used a commercially available program that is widely used around the country. I know of three companies that use this program routinely for dynamic analysis. One of these was the contractor involved in the blast test that I spoke of, another is a large company in the St. Louis area, and the third is a very large company not very many miles south of here. I have observed this and I am pretty sure of my ground.

Mr. Paz, (University of Louisville): I don't know if the problem I am going to present is directly related to yours but we also had trouble with a consistent mass matrix some years ago. I suggested that a student who was working on a master's thesis compare results using the consistent mass, a lumped mass, and also what might be called the exact solution of the Bernoulli-Euler equations. He found tremendous differences. To test the programs he used simple numbers like units for everything, such as the modulus of elasticity and he found tremendous differences in computing the consistent mass. So I investigated this further and made a series expansion of the exact solution; I found out that the consistent mass is the first two terms of the series expansion, where one term is the stiffness and the other term would be the mass that we used in the consistent. By using these simple numbers to test the programs he was out of the range of convergence of the series so this explained why the consistent mass gave wrong answers. This might not be your case because you checked this with another program, it is probably a question of programming. But I thought it might be of interest to know that because with consistent mass we have to be careful in the convergence of the series that actually represents these terms.

Mr. Mains: One of the things I try very hard to do with my classes is to make sure that the students understand what they are doing. So I make them go through dynamic analysis the hard way with the hands-on operation of the programs, so that they know what is going on at each step and so that they come out of it with a set of solved problems that they can use to check out any black box they are subjected to later on. When they go out from the University to the job, quite routinely they are handed a problem and told to put this on the computer and get some answers. They have no opportunity to find out what the computer has on it. I think this is a fairly widespread practice I get feed back from it. Every student that goes out and then comes back to visit a year or two later tells me the same story.

RESEARCH METHOD OF THE EIGENMODES AND GENERALIZED ELEMENTS OF A LINEAR MECHANICAL STRUCTURE

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The determination of the eigenfrequencies, eigenmodes and generalized elements of a structure is fundamental in the study of its dynamic behavior (e.g. fluttering of planes).
Of all the methods tested, those based on the appropriation of modes seem to give the most accurate results.
The experimental methods used today are often unreliable and do not always guarantee that all the eigenmodes corresponding to a given frequency range have been isolated. The method which we suggest does not present these drawbracks. This method is based on the appropriation method and permits to determine the eigenvectors and generalized elements directly by calculus from the forced responses to a given frequency.

I - DEFINITION OF APPROPRIATION

Appropriation consists of finding which system of forces must be applied to a structure in order to obtain :

- a response proportional to an eigenmode of the associated conservative system ;
- the eigenfrequency corresponding to this mode.

Most experimental methods are based on the fact that all points of the structure have the same phase angle when the excitation is appropriate.

In the simplest methods, the experimenter proceeds tentatively by modifying the level of excitating forces and the frequency as best he can in order to minimise the force and velocity phase difference for all points. A method based on this principle was tested by LEWIS and WRISLAY [1] at the M.I.T. in 1950.

A. DECK. [2] from the ONERA has developed an automatic method for appropriation which proceeds by successive approximations assuming that frequencies have been isolated. Basically the minimum of the in phase response relative to the excitation is detected by varying each applied force and the excitation frequency successively. This method offers the advantage of being well-suited to automation but it is unsuccessful whenever the eigenfrequencies are too close to one another.

With the method developed by D. CLERC [3] appropriate forces can be calculated directly from a set of p responses at a given frequency related to p linearly independent excitation configurations. Such a method is systematic, it takes the information from all pickups into account and two close modes or more can easily be detected, but a great amount of measurements and calculus is required to determine one eigenmode.

TRIAL-NASH [4] had already used a less sophisticated form of this idea as early as 1958.

J.J. ANGELINI [5] introduced the matrix $^{T}V.F$ calculated from the real part V of p responses to p

linearly independent excitation configurations stored in the matrix F. He takes advantage of the vanishing property of the ${}^{T}VF$ determinant when the excitation frequency is similar to the eigenfrequency. Such a method requires a great deal number of measurements and only takes account of displacements at the excitation points. Such methods have obvious drawbacks resulting either from exploitation (number of measurements required) or lack of efficiency (eigenmodes undetected in the case of close frequencies).

The aim of this paper is to develop a process allowing all the eigenmodes near the excitation frequency to be located, then secured from a limited set of measurements.

11 - Preliminary considerations

In forced harmonic conditions, the movement of a discrete linear system is defined by the equation :

$$M\bar{y}^{\circ\circ} + B\bar{y}^{\circ} + K\bar{y} = f \cdot e^{j\omega t} , \qquad (1)$$

M, B, K being the mass, damping and stiffness matrices respectively.

$\bar{y} = y\, e^{j\omega t}$, leads to

$$(K - \omega^2 M + j\omega B)\, y = f \qquad (2)$$

To the damped initial system, we can associate the conservative system defined by the eigenproblem

$$(K - \lambda M)\, y = 0 , \qquad \lambda = \omega^2 , \qquad (3)$$

from which we deduce :

- the modal matrix Y of eigenvectors y_ν
- the spectral matrix Λ of eigenvalues λ_ν

According as the eigenvectors are arbitrarily normed (z_ν) or relatively to the mass matrix (y_ν), we obtain the classical relations :

$${}^{T}YMY=E, \quad {}^{T}YKY=\Lambda, \quad {}^{T}ZMZ=\mu , \quad {}^{T}ZKZ= \mu \cdot \Lambda = \gamma \qquad (4)$$

where μ is the diagonal matrix of the generalized masses, γ that of the generalized stiffness.

The forced displacements of the damped system can be decomposed on the basis of the eigenvectors y_ν or z_ν of the associated conservative system. Then, we obtain :

$$y(\lambda)=Y\Omega^{T}Y.f \quad \text{or} \quad y(\lambda)=Z\tilde{\Omega}^{T}Z.f \qquad (5)$$

$$\text{with } \Omega =(\Lambda-\lambda E+j\sqrt{\lambda}\beta)^{-1} \text{ or } \tilde{\Omega} =(\gamma -\lambda \mu +j\sqrt{\lambda}b)^{-1} \qquad (6)$$

$$\beta = {}^{T}YBY \qquad b={}^{T}ZBZ= \mu^{1/2}\beta\,\mu^{1/2} \qquad (7)$$

where β and b are the normed and non-normed generalized damping matrices.

12 - Determination of appropriate forces

It is known [6] that a linear self-adjoint system can be appropriate to an eigen circular frequency ω_ν with a system of forces all in phase (or anti phase). For exemple, at the circular frequency ω_ν , the movement is described by :

$$(K-\lambda_\nu M+j\sqrt{\lambda_\nu}B)z=f$$

if $z=Z \cdot e_\nu = z_\nu$ $(K-\lambda_\nu M)z_\nu = 0$ then

$$j\sqrt{\lambda_\nu}BZ.e_\nu=f_\nu \quad \text{or still}$$

$$j\sqrt{\lambda_\nu}\,{}^{T}ZBZ\, e_\nu = j\sqrt{\lambda_\nu}b\, e_\nu={}^{T}Z \cdot f_\nu$$

$$f_\nu=j\sqrt{\lambda_\nu}\,{}^{T}Z^{-1}.b.e_\nu=j\sqrt{\lambda_\nu}\,{}^{T}Y^{-1}\beta e_\nu \qquad (8)$$

If b is diagonal (Basile's hypothesis verified [7]), this force f_ν will excite an eigenmode which is out of phase in relation to the excitation whatever the frequency.

Let $z = Z \cdot c$

At the circular frequency $\omega=\sqrt{\lambda}$, the response z to the appropriate excitation will be defined by :

$$(K-\lambda M+j\sqrt{\lambda}B)z = f_\nu$$

hence :

$${}^{T}Z(K-\lambda M+j\sqrt{\lambda}B)Z.c = j\sqrt{\lambda_\nu}b\, e_\nu$$

$$(\gamma -\lambda \mu +j\sqrt{\lambda}b).c= j\sqrt{\lambda_\nu}b\, e_\nu$$

$$c = j\sqrt{\lambda_\nu}\,\frac{b_{\nu\nu}}{\gamma_{\nu\nu} - \lambda\mu_{\nu\nu} + j\sqrt{\lambda}\, b_{\nu\nu}}\, e_\nu$$

All the points of the structure vibrate in phase. The phase difference ϕ between the movement and excitation force is defined by :

$$\operatorname{tg}\phi = \frac{\sqrt{\lambda}.b_{\nu\nu}}{\lambda\mu_{\nu\nu}-\gamma_{\nu\nu}} = \frac{\sqrt{\lambda}.\beta_{\nu\nu}}{\lambda-\lambda_\nu} \qquad (9)$$

Reciprocally, assuming that a real force vector f is found, whose response to any circular frequency is

$$y = Y\Omega^{T}Y \,.\, f = Vf + j\,f$$

with $V \,.\, f = k\, W \,.\, f$, and k = real constant,

then f is an appropriate force to an eigenvector. Indeed we have :

$V = Y\, R\,(\Omega)\, {}^{T}Y$ $\quad$ R (Ω) real part of Ω

$W = Y\, J\,(\Omega)\, {}^{T}Y$ $\quad$ J (Ω) imaginary part of Ω.

Ω diagonal matrix of general term

$$\frac{\lambda_\nu-\lambda-j\sqrt{\lambda}\beta_{\nu\nu}}{(\lambda_\nu-\lambda)^2+\lambda\beta_{\nu\nu}^2}$$

The above equality V.f=k.W.f leads to

$$Y\left[\frac{\lambda_\nu-\lambda}{D}\cdot\right]{}^{T}Y.f = kY\left[-\frac{\sqrt{\lambda}\beta_{\nu\nu}}{D}\right]{}^{T}Y.f$$

$$D = (\lambda_\nu-\lambda)^2 + \lambda\beta_{\nu\nu}^2 \,.$$

Multiplying this equation by $\left[-\frac{D}{\sqrt{\lambda}\cdot\beta_{\nu\nu}}\right].\, Y^{-1}$ we obtain :

$$\left[\left[\frac{\lambda-\lambda_\nu}{\sqrt{\lambda}\beta_{\nu\nu}}\right] - k\,E\right]{}^{T}Y.f = 0 \qquad (10)$$

Therefore there is an eigenvalue λ_ν such that

$$k = \frac{\lambda-\lambda_\nu}{\sqrt{\lambda}.\beta_{\nu\nu}}$$

The corresponding eigenvector is of the form

$${}^{T}Y.f = \begin{vmatrix} 0 \\ 0 \\ . \\ \alpha \\ . \\ 0 \end{vmatrix}$$

Therefore, f is proportional to the appropriate force to the ν^{th} eigenmode. For instance f_ν is defined by :

$$f_\nu = MY\sqrt{\lambda}\begin{vmatrix} 0 \\ 0 \\ . \\ \beta_{\nu\nu} \\ . \\ 0 \end{vmatrix} \quad \text{which leads to} \quad {}^{T}Y.f_\nu = \sqrt{\lambda}\begin{vmatrix} 0 \\ 0 \\ . \\ \beta_{\nu\nu} \\ . \\ 0 \end{vmatrix}$$

Such result multiplied by an arbitrary constant is the same as equation (8) because ${}^{T}Y^{-1} = MY$.

Consequently, if a force f can be found so that V.f = kW.f, this force is appropriate to an eigenmode of the structure, and the displacement y is proportional to this eigenmode. This property constitute the basis of the appropriation criterion suggested.

k can be a double eigenvalue :

$$k = \frac{\lambda-\lambda_\nu}{\sqrt{\lambda}\,\beta_{\nu\nu}} = \frac{\lambda-\lambda_\sigma}{\sqrt{\lambda}\,\beta_{\sigma\sigma}} \,.$$

In that case two linearly independent eigenvectors will correspond to this double value of k.

$${}^{T}Y.f_\nu = \begin{vmatrix} 0 \\ 0 \\ . \\ \alpha_\nu \\ . \\ 0 \end{vmatrix} \quad \text{and} \quad {}^{T}Y \,.\, f_\sigma = \begin{vmatrix} 0 \\ 0 \\ . \\ . \\ \alpha_\sigma \\ 0 \end{vmatrix}$$

The force f found will then be f_ν or f_σ or more generally a linear combination of both. Such indetermination can always be erased by selecting another excitation frequency.

13 - Appropriation criterion when Basile's hypothesis is verified

p excitation configurations linearly independant can be applied to the structure in p points at a given ω circular frequency and the responses in phase and quadrature with the excitation in n points be noted. The responses in phase and in quadrature as well as the forces can be rearranged under matrix form resulting in matrices V(n,p), W(n,p) and F(p,p) respectively. We can assume for instance F=E, E being the uni-

ty matrix.

With such linear system, a response to any vector force "a" will be Va + jWa. An appropriate force to an eigenmode will be the vector "a_ν" such that :

$$V.a_\nu = k\, W\, a_\nu \qquad (11)$$

In the case when the matrix of applied forces $\tilde{F}$ is not a unity matrix, we can measure responses in phase and quadrature $\tilde{V}$ and $\tilde{W}$ where each column is a linear combination of the V and W columns. We can therefore write them under the form :

$\tilde{V} = V \,.\, \tilde{F}$ and $\tilde{W} = W \,.\, \tilde{F}$.

We try to find the values of k such that :

$\tilde{V}\,\tilde{a}_\nu = k\,\tilde{W}\,\tilde{a}_\nu$ for instance

$V\,\tilde{F}\,\tilde{a}_\nu = k\,W\,\tilde{F}\,\tilde{a}_\nu$

An appropriate force in this case is equivalent to $\tilde{F}.\tilde{a}_\nu$. For simplicity, we now assume that

$\tilde{F} = F = E$

In practice, we strictly cannot guarantee equality (11). We try to find the vector "a" and the scalar k such that V.a - kWa = ε is minimum.

Let ${}^T\varepsilon\varepsilon = {}^Ta\,({}^TV - k\,{}^TW)\,(V - kW)\,a$

To minimize ${}^T\varepsilon\varepsilon$ it is necessary to compare ${}^T\varepsilon\varepsilon$ to a norm taking into account the amplitude of the movement. We are led to minimize the parameter δ .

$$\delta = \frac{{}^T\varepsilon\varepsilon}{\|Va\| + \|Wa\|} = \frac{{}^Ta({}^TV - k\,{}^TW)(V - kW)a}{{}^Ta({}^TVV + {}^TWW)a}$$

The extremes of δ are given by the zero of the partial derivatives $\frac{\partial\delta}{\partial a_i} = 0$, i=1,...p and $\frac{\partial\delta}{\partial k} = 0$ which leads to

$$[{}^TVV - k({}^TVW + {}^TWV) + k^2\,{}^TWW - \delta({}^TVV + {}^TWW)]\,a = 0 \qquad (12)$$

$$k = \frac{{}^Ta\,{}^TWVa}{{}^Ta\,{}^TWWa} \qquad (13)$$

(12) is an eigenvalue problem in δ non-linear in relation to the parameter k. It suffices to find which are the values k_ν of k making the smallest eigenvalue δ minimum. To each k_ν there is a corresponding eigenvector "a_ν" which is the appropriate force to the ν^{th} eigenmode. We check that the equation (13) is satisfied for each solution found.

14 - Form of the appropriate forces such that V.a = kW.a when Basile's hypothesis is not verified

In this case in order to obtain V.a=kW.a at $\lambda \neq \lambda_\nu$, complex forces $F_R + jF_i$ must be applied. We can show that the generalized forces appropriated to the ν^{th} eigenmode defined by

$${}^TZ(F_R + jF_i) = f_R + j\,f_i$$

are of the form :

$$f_R = \begin{vmatrix} -\sqrt{\lambda}b_{1\nu} \\ -\sqrt{\lambda}b_{2\nu} \\ \cdot \\ \cdot \\ -\sqrt{\lambda}b_{\nu\nu} + k(\lambda_{\nu\nu} - \lambda\mu_{\nu\nu}) \\ \cdot \\ \cdot \\ -\sqrt{\lambda}b_{n\nu} \end{vmatrix} \quad f_i = \begin{vmatrix} k\sqrt{\lambda}b_{1\nu} \\ k\sqrt{\lambda}b_{2\nu} \\ \cdot \\ \cdot \\ k\sqrt{\lambda}b_{\nu\nu} + (\gamma_{\nu\nu} - \lambda\mu_{\nu\nu}) \\ \cdot \\ \cdot \\ k\sqrt{\lambda}b_{n\nu} \end{vmatrix} \qquad (14)$$

k can be choosen such that f_i is null when b is diagonal. If for instance

$$k = \frac{\lambda\mu_{\nu\nu} - \gamma_{\nu\nu}}{\sqrt{\lambda}\, b_{\nu\nu}} = \frac{\lambda - \lambda_\nu}{\sqrt{\lambda}\, \beta_{\nu\nu}} \qquad (15)$$

The generalized forces are then written :

$$f_R = -\sqrt{\lambda} \begin{vmatrix} b_{1\nu} \\ b_{2\nu} \\ \cdot \\ \cdot \\ \cdot \\ b_{\nu\nu}(1+k^2) \\ \cdot \\ \cdot \\ b_{n\nu} \end{vmatrix} \quad f_i = k\sqrt{\lambda} \begin{vmatrix} b_{1\nu} \\ b_{2\nu} \\ \cdot \\ \cdot \\ \cdot \\ 0 \\ \cdot \\ \cdot \\ b_{n\nu} \end{vmatrix} \qquad (16)$$

In practice $b_{i\nu}$ $(i \neq \nu) << b_{\nu\nu}$, furthermore in the proximity of $\lambda \# \lambda_\nu$, $k \# 0$. Consequently, the imaginary force is negligible compared to the real force, and the appropriation criterion suggested may be applied at the proximity of $\lambda = \lambda_\nu$ even when Basile's hypothesis is not verified.

II - PRACTICAL CALCULATION OF APPROPRIATE FORCES

21 - $\delta = f(k)$ curve plotting.

The calculation is carried out as follows [8] : Given a k_o value of k (for example $k_o = 0$), the values of δ solutions of the eigenproblem (12) and the corresponding eigenvectors are deduced. Then by increasing k and plotting $\delta = f(k)$. We get the diagram : (Fig.1)

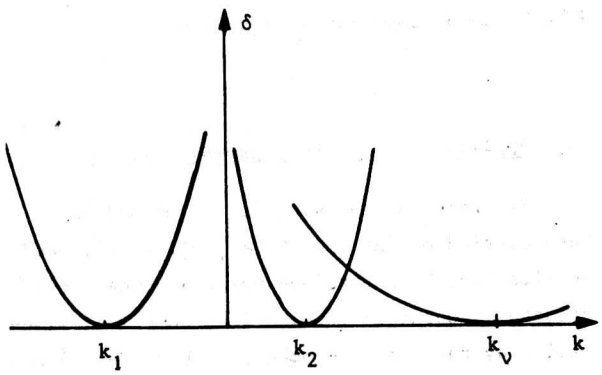

FIG.1 Curves $\delta = f(k)$

For each value k_ν of k with δ is minimum, there is a corresponding force vector "a_ν" appropriate to an eigenmode of the structure. Wa_ν is proportional to the eigenmode.

The appropriation to a mode is all the more accurate as the corresponding value of δ is smaller.

Such calculation is comparatively long and entails a lot of iterations to obtain a k_ν with δ minimum.

A quicker alternative can be summarized as follows : given an arbitrary initial values of k, the eigenproblem (12) can be solved and a new value k'_o of k calculated from the eigenvector obtained using the relation (13). The value k'_o obtained is very close to a k_ν value with δ minimum. The value k can be obtained rapidly through iteration.

In practice, two or three iterations are sufficient to obtain fairly accurate values of k_ν.

The method still gives good results when two eigenfrequencies or more are close or equal. Two k values or more are found to which two appropriate forces, or more, are associated.

The relation $k = \dfrac{\lambda - \lambda_\nu}{\sqrt{\lambda}\beta_{\nu\nu}}$ shows that even when two eigenvalues λ_σ and λ_ν are equal, we still have two separate values for k if $\beta_{\nu\nu} \neq \beta_{\sigma\sigma}$. Any combination of the two obtained vectors is an eigenvector. The two determined eigenvectors correspond to the two modes for which the damping coefficient $\beta_{\sigma\nu}$ is minimum. The above relation also infers that it is possible to obtain a double value for k even if $\lambda_\sigma \neq \lambda_\nu$. In this case, two separate values for k are obtained by changing the excitation frequency.

III - CALCULATION OF MASSES AND GENERALIZED DAMPINGS

The circular frequency $\omega_\nu = \sqrt{\lambda_\nu}$, generalized mass $\mu_{\nu\nu}$ an the damping β have to be determined for each eigenmode.

For that, after locating a frequency and calculating the appropriate corresponding force a_ν, we apply this force to the structure. A step to step predetermined frequency micro-sweeping device records the system complex responses:

$$y(\omega) = y_R(\omega) + j\, y_i(\omega) .$$

31 - Eigen circular frequency ω_ν and generalized damping $\beta_{\nu\nu}$ determination.

Using the following method to determine ω_ν

and $\beta_{\nu\nu}$ we obtained fairly satisfactory results.

For a set of value of $\lambda \neq \lambda_\nu$, the appropriate force a_ν is applied and

$$k(\lambda) = \frac{{}^T y_R \cdot y_i}{{}^T y_i \cdot y_i} = \frac{\lambda - \lambda_\nu}{\sqrt{\lambda} \cdot \beta_{\nu\nu}} \quad \text{is plotted}$$

Then $k(\lambda) = 0$ gives $\lambda = \lambda_\nu$ and the slope of the curve $k(\lambda)$ gives the generalized damping $\beta_{\nu\nu}$:

$$\left(\frac{dk}{d\lambda}\right)_{\lambda=\lambda_\nu} = \frac{1}{\sqrt{\lambda_\nu}.\beta_{\nu\nu}} \quad \text{hence} \quad \beta_{\nu\nu} = \frac{1}{\sqrt{\lambda_\nu}\left(\frac{dk}{d\lambda}\right)_{\lambda=\lambda_\nu}} \qquad (17)$$

32 - Generalized mass determination

Generalized mass and damping can be obtained by the diagram of complex power [9] P_R and P_i as a function of λ .

$$P_R = {}^T a_\nu \cdot y_R \qquad P_i = {}^T a_\nu \cdot y_i$$

For $\lambda \neq \lambda_\nu$, it is known |4| that :

$$P_R = (\lambda_\nu - \lambda)\mu_{\nu\nu} \qquad (18)$$

$$P_i = (2\sqrt{\lambda_\nu} - \sqrt{\lambda})\, b_{\nu\nu}. \qquad (19)$$

Hence for $\lambda = \lambda_\nu$,

$$P_R = 0 \ , \quad \frac{dP_R}{d\lambda} = -\mu_{\nu\nu}, \qquad (20)$$

$$P_i = -b_{\nu\nu}\sqrt{\lambda} = -\mu_{\nu\nu}\beta_{\nu\nu}\sqrt{\lambda_\nu} \quad \text{from which}$$

$$\mu_{\nu\nu} = -\frac{P_i}{\sqrt{\lambda_\nu}\ \beta_{\nu\nu}} \qquad (21)$$

A simple geometric diagram allows to check the coherence of the results obtained (fig.2).

The experiment shows that :

- The curve $k(\lambda)$ (17) provides accurate values for λ_ν and $\beta_{\nu\nu}$ since all the points where pickups have been placed are taken into account.

- The values of $\mu_{\nu\nu}$ and λ_ν given by (20) are weighted with errors which are due to the fact that $\frac{dP_R}{d\lambda}$ varies very quickly in the proximity of $\lambda = \lambda_\nu$;

- The extreme of P_i is slightly modified due to error effects in measurements and (21) enables one to obtain $\mu_{\nu\nu}$ accurately.

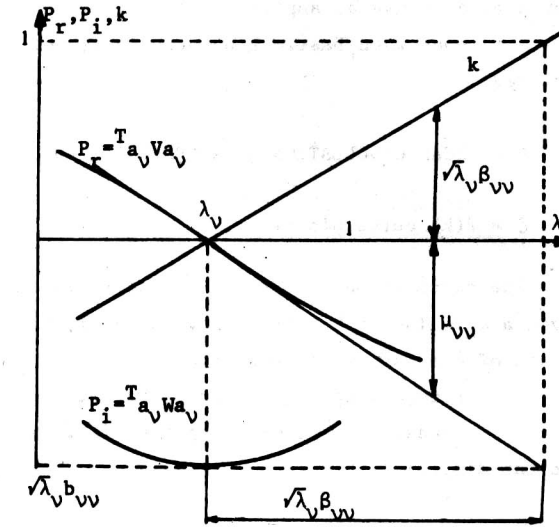

FIG.2 Curves $k(\lambda)$ and complex power

33 - Applications of the suggested method

Experimental checking of the method has been carried out at the ONERA [10]. Fairly good results have been obtained using a plane substructure. Obtained results from numerical simulation by the finite elements method of a system having close natural frequencies, are given here. This system is a clamped, free beam folded up as shown Fig.3. It can be shown that such a structure evinces a succession of eigen-frequencies by pairs.

The eigenfrequencies in each pair are all the closer as $l_1 - l_2$ gets smaller

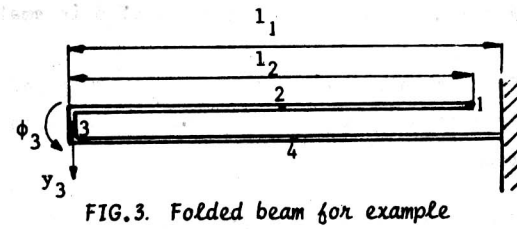

FIG.3. Folded beam for example

The first four eigen-frequencies of the

system are :

F_1=13.1 Hz F_2=13.3Hz F_3=119Hz F_4=120.8Hz

In the finite element method, three degrees of freedom are associated to each mode (transverse, longitudinal and angular displacements) which leads to a 12 degrees freedom system when four elements are concerned.

In the generalized damping matrix introduced the value of the quality factor $Q=\omega_\nu/\beta_{\nu\nu}$ of all modes is 20 or about and coupled termes are added between the first three modes.

The submatrix β is represented below

$$\beta = \begin{bmatrix} 4.05 & -0.54 & -0.27 \\ -0.54 & 3.24 & -0.81 \\ -0.27 & -0.81 & 27 \end{bmatrix}$$

The simulated testing was carried out as follow : at a given frequency, responses to forces equal to 1 applied successively to each transverse degree of freedom are computed. The responses are stored into the V and W matrices from which $k = f(\lambda)$ is plotted.

The following graph is obtained by selecting an excitation frequency between the first two eigenfrequencies (f = 13.2 Hz)

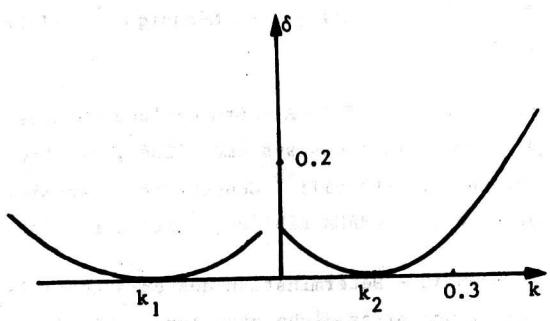

FIG.4 Curves δ = F(k)

The two parabolic curves $\delta = f(k)$ clearly show that the system has two eigenfrequencies located on both sides of the excitation frequency. Then the appropriate forces to the two modes can be computed.

Force a_1 appropriate to the first eigenmode is computed at an excitation frequency equal to 12.13 Hz. A microsweeping from f=11.9Hz to f=12.3 Hz gives the following curves (fig.5) for k, ^{T}aVa, ^{T}aWa.

Similary, force a_2 appropriate to the second eigenmode is computed at an excitation frequency equal to 13.27 Hz. A microsweeping from f=13.1 to f=13.5 Hz gives the results in fig.5.

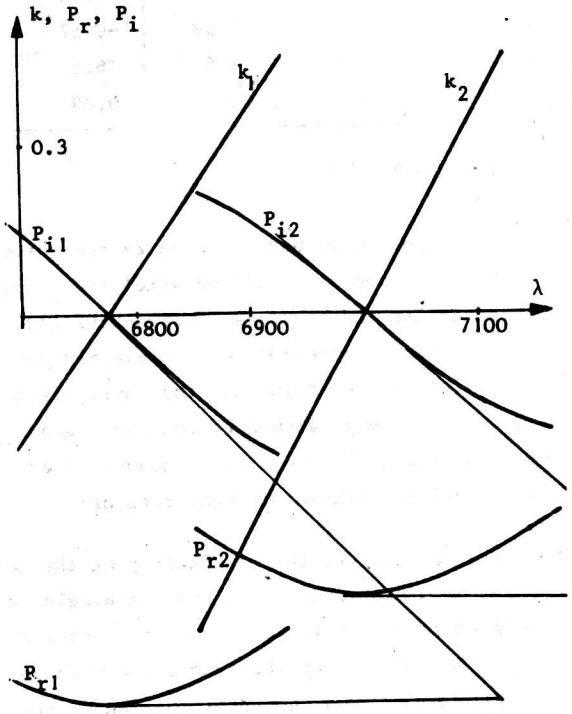

FIG.5 k(λ) and complex power. Modes 1 and 2

The generalized parameters of eigenmodes are obtained from relations (17) and (21). The following table allows the computed results to be compared with the exact values. ω, β, μ, y_i, ϕ_i, are the circular frequency, the generalized damping and mass the displacement and the rotation of the ith cross-section. It can be seen that results are obtained with good accuracy though the two modes are very close and strongly coupled. It has also been verified that correct results can be secured when error effects in mea-

surements are simulated.

	First Mode		Second Mode	
	Exact Values	Computed Values	Exact Values	Computed values
ω^2	6775	6775	7000	7000
β	4.05	4.17	3.24	3.26
μ	$2.05\ 10^{-3}$	2.10^{-3}	0.181	0.183
y_1	-11.4	-11.1	332	332
ϕ_1	0.130	0.128	-1.39	-1.37
y_2	7.8	7.91	131	134
ϕ_2	0.127	0.126	-1.27	-1.25
y_3	26.2	26.2	-16.1	-12.4
ϕ_3	0.123	0.122	-0.59	-0.57
y_4	8.63	8.65	14.6	16.6
ϕ_4	0.1	0.1	0.07	0.09

IV - CONCLUSION

The suggested method has been tested twice on a numerical computer, adding simulated errors of measurements and on plane substructure [10]. Even when two or more neighbouring eigenfrequencies are very closed together, the appropriate force can be easely calculated and the eigenmodes restored with a negligible amount of errors compatible with measurement accurary.

As a conclusion, the originality of the method can be summarised as follows : a single set of measurements with given frequencies enables one to determine the appropriate force with several eigenmodes, and the determination of the values of k such that δ is minimum enables one to locate these eigenfrequencies ;

- multiplie or neighbouring are automatically detected even though eigenmodes are strongly coupled by damping ;

- if we agree to make a few measurements in the proximity of an eigenfrequency, the diagram in function of λ of k, P_R and P_i enables one to determine :

- the eigen-circular frequency ω_ν ;
- the generalized damping $\beta_{\nu\nu}$;
- the generalized mass $\mu_{\nu\nu}$.

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CALCULATION OF NATURAL FREQUENCIES AND MODE SHAPES OF MASS LOADED AIRCRAFT STRUCTURES

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Aircraft optical packages are forced by the local random vibration response characteristics of the airframe. Hence, the vibration characteristics must be known by the designer of optical packages. However, since the environment changes with the addition of the electro-optical system, it is necessary to predict the modified vibration environment, either by flight testing with mass simulations or conducting structural analysis. This paper poses the problem of loaded random vibration response estimation using Galerkin's method, a direct method, and a generalized coordinates approach. Results indicate that both the direct method and the generalized coordinates solution give very good estimates of the first four modes of a mass loaded simply supported beam, with the generalized coordinates solution giving better accuracy than the direct method. Galerkin's method gives unacceptable results. In addition, a matrix iteration scheme for computing loaded modes, given unloaded modes, is presented. Results show that over 50% reduction in execution time is possible in computing the first four modes, with good accuracy maintained.

INTRODUCTION

In the design of airborne optical packages, the angular and rectilinear vibration response characteristics of the airframe represent the input forcing functions to the optical systems [1]. Hence, it is necessary to collect vibration data for the aircraft structure on which the optical system is to be mounted. The problem with such data is that installing the optical system in the aircraft may change the airframe vibration response characteristics. Therefore, it is desirable to collect aircraft data with a mass simulation installed. However, when the flight test is conducted, the optical system design may not be finalized, so the most appropriate simulation mass may not be known. In addition, unloaded data may already be available for the aircraft in which case it is more desirable to use existing data than increase cost by collecting additional data. A better approach would be to calculate or estimate the mass loaded random response, using some relatively simple procedure, rather than to rely on installing mass simulations and then conducting flight tests. This paper is an analysis of four procedures for calculating the mass loaded vibration response of aircraft structures.

Finite element analysis is frequently used to compute the loaded random vibration response of aircraft structures. However, unless a finite element model already exists, such a process would be time consuming. Even if a finite element model did exist, the manner of mass loading would have to be included and possible matrix size reduction techniques implemented. In addition, there are persistent problems concerning the choice of appropriate forcing functions used as inputs. Thus, finite element modeling might be more expensive and time consuming in some instances than conducting flight tests.

Three analytical techniques for computing the mass loaded response of aircraft structures are compared to the finite element answer. Those analytical techniques are: Galerkin's method, a generalized coordinates approach, and a direct method based on Hamilton's law of varying action. In addition, an alternative method of computing the loaded response using finite element analysis and knowing the unloaded response is presented.

ANALYSIS

Approximations to the solutions of