ELEMENTS OF COMPLEX ANALYSIS

SONNENSCHEIN

GREEN

Elements of Complex Analysis

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Preface

This book is the result of lectures given by Prof. Dr. J. Sonnenschein at the Brussels University. It is an introductory course in complex analysis of one complex variable. It contains sufficient material for about thirty lectures.

The aim of the book is to introduce the principal notions and theorems of complex analysis and to make the reader acquainted as quickly as possible and with as much rigor as can be obtained in a short course, with the knowledge necessary to use the most important results of complex analysis in pure and applied mathematics.

An introductory course to complex analysis is not supposed to contain original results. The only original features of this book lie in the presentation of the material.

Chapter 0 introduces the reader to the basic notions of topology which are necessary to understand the following chapters. Those students who have followed a course of topology may skip Chapter 0.

The following four chapters present the essentials of complex analysis. It seems to us that four notions form the basis of complex analysis. In each of the chapters 1 to 4 we have treated one of these notions: In the first chapter the notion of complex differentiability of a function, in the second one the property of a function that its integral over any closed curve of a certain family of curves is zero. This property we called holomorphicity. In the third chapter we have introduced the notion "analytic" for functions which possess about any point of a domain an expansion in a Taylor series. The fourth property of functions, which is studied in the fourth chapter, is a geometric property, namely the conformality of the mapping.

These four notions which by their definition are quite different are shown to imply each other, that is, to be equivalent.

In our definitions and proofs we have been influenced by many books on the same subject, especially by the books of L. Ahlfors: *Complex*

Analysis; H. Cartan: *Théorie élémentaire des fonctions analytiques d'une ou plusieurs variables complexes;* Z. Nehari: *Conformal mapping;* and others.

It is a pleasure for us to express our gratitude to Prof. Dr. J. P. Gossez of the Brussels University for his suggestions and valuable remarks and to Profs. Dr. Emil Herzog and Dr. Ta Li of California State Polytechnic University who checked the manuscript and prepared the solutions for the extensive Problem Section and the Teacher's Manual which will form a valuable addition to the text.

Jacob Sonnenschein
Simon Green

Contents

Preface *vii*

chapter 0

Algebraic and Topological Preliminaries

1

§1. Some notions of set theory *1*
  1. Logical symbols *1*
  2. Sets *1*
  3. Functions *5*
  4. Sequences and countable sets *6*

§2. Sets with an algebraic structure *9*
  1. Groups *9*
  2. Rings and fields *10*
  3. Vector spaces *11*

§3. Sets with a topological structure—topological spaces *12*
  1. Definition of a topology *12*
  2. Continuous functions *15*
  3. Connectedness and the connectedness argument *17*
  4. Compactness *20*

§4. Metric spaces *22*
  1. Definitions *22*
  2. Normed vector spaces *24*
  3. Sequences in metric spaces; completeness *25*
  4. Continuity in metric spaces *28*
  5. Compactness in metric spaces *30*

chapter 1

Complex Differentiable Functions

37

§1. Summary of the course *37*

§2. The complex numbers *38*

1. Algebraic aspects *38*
2. Geometric representation of complex numbers *39*
3. Arguments of products and quotients *44*
4. The Riemann sphere *47*

§3. Topology of the complex plane *51*

1. Neighborhoods and discs in the complex plane; open and closed sets *51*
2. Sequences in the complex plane *52*
3. Compact sets in the complex plane *53*
4. Connected sets and domains in the complex plane *55*
5. Arcs and closed curves *58*

§4. Differentiable functions *60*

1. Continuous functions *60*
2. Differentiability *63*
3. The Cauchy-Riemann equations *66*
4. Primitives *70*
5. Geometric interpretation of the derivative *71*
6. Isomorphisms *73*

chapter 2

Holomorphic Functions

77

§1. Homotopy and line integrals *77*

1. Homotopy of closed curves with respect to a domain D *77*
2. Line integrals *81*
3. Properties of line integrals *86*

§2. Cauchy's theorem *93*

1. Cauchy's theorem for starlike domains *93*
2. Cauchy's theorem for general domains; residues *99*

§3. Cauchy's formula *106*

1. The multivalued function log z *106*
2. Index of closed curve *110*

3. Linear integral transformations *112*
4. Cauchy's formula *116*

§4. Compactness of a bounded and closed subset of $H(D)$ *120*

§5. Harmonic functions *126*

§6. The mean-value property and maximum modulus principle *129*

chapter 3

Analytic Functions

133

§1. Power series *133*

1. Numerical series *133*
2. Series of functions and uniform convergence *136*
3. Power series *137*
4. A function holomorphic on D is analytic on D *142*
5. Cauchy's inequalities *145*
6. Schwarz's lemma *146*
7. Zeros of an analytic function *146*
8. Analytic continuation *149*
9. Permanency principle of functional relations *153*

§2. Laurent series *156*

1. Definition of Laurent series *156*
2. Isolated singularities and their classification *160*
3. Meromorphic functions *164*
4. The point at infinity *165*

§3. Residues and their applications *169*

1. The residue theorem *169*
2. Applications of the residue theorem *172*
3. The logarithmic residue *183*

chapter 4

Conformal Mappings

191

§1. Conformal mappings *191*

1. Definitions and relations *191*
2. Mapping at points where $f' = 0$ *194*
3. The open mapping theorem *196*

§2. Homographies *199*
1. Definitions and basic properties *199*
2. Symmetry *204*
§3. Riemann's mapping theorem *208*

Solutions to the Odd-Numbered Problems *213*

Index *279*

chapter 0

Algebraic and Topological Preliminaries

§1. Some notions of set theory

1. Logical symbols

We introduce here some logical symbols which we shall use throughout this book.

Let S_1 and S_2 be two statements, then the symbol $S_1 \Rightarrow S_2$ means that statement S_1 implies statement S_2. The symbol $S_1 \Leftrightarrow S_2$ means $S_1 \Rightarrow S_2$ and $S_2 \Rightarrow S_1$, and we say that statement S_1 is equivalent to statement S_2.

Another way to express $S_1 \Leftrightarrow S_2$ is to say S_1 holds if and only if S_2 holds, or, briefly S_1 holds *iff* S_2 holds. [*Iff*, with double *f*, is an abbreviation for "if and only if."]

The symbol $\exists$ means "there exists," and the symbol $\ni$ stands for "such that."

EXAMPLE

"$\exists$ at least one integer $x \ni \frac{3}{4} < x < \frac{5}{2}$" means "there exists at least one integer x such that $\frac{3}{4} < x < \frac{5}{2}$."

The symbol $\forall$, an inverted A, means "for all."

2. Sets

We assume that the reader of this book is acquainted with elementary set theory; in any case, we remind him of the following notations:

$a \in A$, the element a belongs to the set A.

$a \notin A$, the element a does not belong to the set A.

$A \subset B$ or $B \supset A$, A is a subset of B; that is, $a \in A \Rightarrow a \in B$.

$A \not\subset B$, A is not a subset of B. We shall write $A = B$ if $A \subset B$ and $B \subset A$.

We shall use the symbol $\varnothing$ for the void or empty set, $\{a\}$ for the set containing only the element a, $\{a_1, a_2, \ldots, a_n\}$ for the set containing the elements $a_1, a_2, a_3, \ldots, a_n$, and $\{a \mid P\}$ for the set of all a having property P.

$A \cup B$ is the union of the sets A and B: $A \cup B = \{a \mid a \in A \text{ or } a \in B\}$. It is the set of all elements a which belong either to A or to B or to both

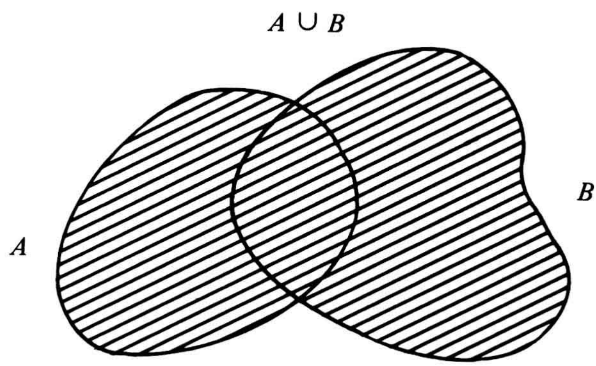


Figure 0.1

of them. $A \cap B$ is the intersection of the sets A and B: $A \cap B = \{a \mid a \in A$ and $a \in B\}$; $A \cap B$ is the set of all elements which belong to A and also to B.

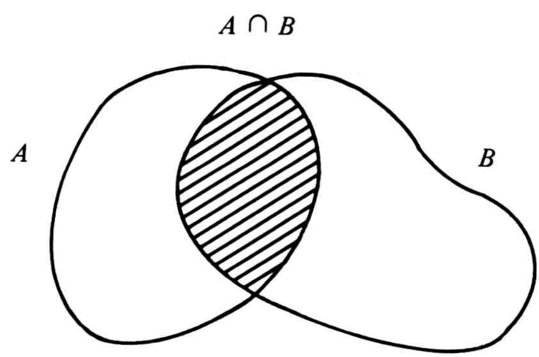


Figure 0.2

If $\mathscr{C}$ is a collection of sets C then the union or intersection of all sets C belonging to $\mathscr{C}$ will be written:

$$\bigcup_{C \in \mathscr{C}} C \quad \text{or} \quad \bigcap_{C \in \mathscr{C}} C$$

By $B \backslash A$ we mean the set of all elements belonging to B which do not belong to A; we do not require $B \supset A$: $B \backslash A = \{a \mid a \in B,\ a \notin A\}$. Clearly $B \backslash A = B \backslash (A \cap B)$.

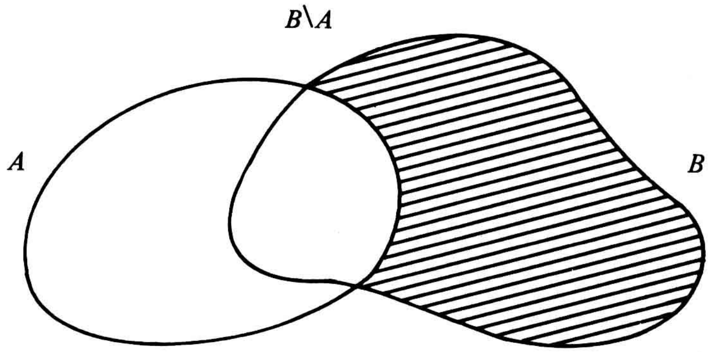


Figure 0.3

Let A, A_1, and A_2 be three sets. We verify easily de Morgan's duality relations:

$$A \backslash (A_1 \cup A_2) = (A \backslash A_1) \cap (A \backslash A_2)$$

and

$$A \backslash (A_1 \cap A_2) = (A \backslash A_1) \cup (A \backslash A_2)$$

Let us prove the first of the two relations: If $x \in A \backslash (A_1 \cup A_2)$, then $x \in A$, $x \notin A_1$ and $x \notin A_2$. Therefore, $x \in A \backslash A_1$ and $x \in A \backslash A_2$, which shows that $x \in (A \backslash A_1) \cap (A \backslash A_2)$. On the other hand, if $x \in (A \backslash A_1) \cap (A \backslash A_2)$, then $x \in A$, $x \notin A_1$ and $x \notin A_2$. It follows that $x \in A$ and $x \notin A_1 \cup A_2$, or $x \in A \backslash (A_1 \cup A_2)$.

The proof of the second duality relation is left to the reader.

DeMorgan's relations may be generalized for an infinite collection $\mathscr{C}$ of sets C in the following way:

$$A \backslash \bigcup_{C \in \mathscr{C}} C = \bigcap_{C \in \mathscr{C}} (A \backslash C)$$

and

$$A \backslash \bigcap_{C \in \mathscr{C}} C = \bigcup_{C \in \mathscr{C}} (A \backslash C)$$

Let A and B be two sets; we define the *Cartesian product* $A \times B$ to be the set of ordered pairs (a, b) with $a \in A$ and $b \in B$:

$$A \times B = \{(a, b) | a \in A, b \in B\}$$

We sometimes say that a is the first component and b the second component of the pair (a, b).

Further on, we shall denote by—

- **Z** the set of integers;
- **P** the set of positive integers, or natural numbers;
- **Q** the set of rational numbers;
- **R** the set of real numbers, $\mathbf{R}^+$ the set of positive real numbers, and $\mathbf{R}^-$ the set of negative real numbers;
- **C** the set of complex numbers.

It follows from the definition of the Cartesian product that $\mathbf{R} \times \mathbf{R} = \mathbf{R}^2$ is the set of pairs (x_1, x_2) with $x_1 \in \mathbf{R}$ and $x_2 \in \mathbf{R}$, and

$$\mathbf{R}^n = \{(x_1, x_2, \ldots, x_n) | x_1, x_2, \ldots, x_n \in \mathbf{R}\}$$

Let A be a set. A subset S of $A \times A$ is called a *relation* in A. For example, if $A = \mathbf{R}$, then $S = \{(a, b) | b = a^2, a \in \mathbf{R}\} \subset A \times A$ is a relation in A.

A relation in A is called an *equivalence relation* if it satisfies the following three conditions:

(i) $(a, a) \in S \quad \forall a \in A$ (*reflexive*)

(ii) $(a, b) \in S \Rightarrow (b, a) \in S$ (*symmetric*)

(iii) (a, b) and $(b, c) \in S \Rightarrow (a, c) \in S$ (*transitive*)

EXAMPLE

The set of pairs $(a, b) \in \mathbf{R} \times \mathbf{R}$ with $b > a$ defines the relation "greater than," but is not an equivalence relation since it is neither reflexive nor symmetric.

EXAMPLE

Let A be the set of all intervals $I = [a, b] \subset \mathbf{R}$, and denote by $L(I)$ the length of I. The set of pairs $(I_1, I_2) \in A \times A$ for which $L(I_1) = L(I_2)$ defines an equivalence relation.

3. *Functions*

Let A, B be two sets. We define a *function*, or *mapping*, $f: A \to B$ (A into B) by associating with each element $a \in A$ one and only one element $b \in B$, which we denote by $b = f(a)$.

A function or mapping f from A into B is thus a set of ordered pairs $(a, b) \in A \times B \ni \forall a \in A\ \exists$ exactly one ordered pair (a, b) in the set with first component a. We call $b = f(a)$ the value of a by f, or we say that a is transformed into b by f. We call the set A of all first components the *domain of definition* of f, and the set of the second components the *image* of A under f, denoted by $f(A)$; clearly $f(A) \subset B$. If the image of A under f is B ($f(A) = B$)—that is, if for each $b \in B\ \exists$ an element a in A with $f(a) = b$—then f is said to be a mapping of A *onto* B.

We say that f is *univalent* if $f(a_1) = f(a_2) \Rightarrow a_1 = a_2$. If f is onto and univalent, then we say f is a *one-to-one correspondence* between A and B, and that A and B are *equivalent* (in the sense of cardinality or set theory). A trivial one-to-one mapping is the identity mapping $A \to A$ with $f(a) = a\ \forall\ a \in A$.

Let f be a one-to-one mapping from A onto B; then to a given $b \in B$ there corresponds one and only one element $a \in A$. This means b is mapped by a certain function into A. We denote this function by f^{-1} and call it the *inverse function*, or *inverse mapping*, of f. Hence, if f is one-to-one, then $f(a) = b \Rightarrow f^{-1}(b) = a$.

If $A_1 \subset A$ and $B_1 \subset B$, then we write $f(A_1) = \{f(a) | a \in A_1\}$ and $f^{-1}(B_1) = \{a | f(a) \in B_1\}$. If f is a mapping $A \to B$ and g is a mapping $B \to C$, then we call the mapping of $A \to C$ which contains all pairs $(a, g(f(a)))$, with $a \in A$, $f(a) \in B$, and $g(f(a)) \in C$, the composition of g and f, denoted by $g \circ f$. Evidently, $\{a, g(f(a))\}$ is a subset of $A \times C$. If f and g are one-to-one, then $g \circ f$ is also one-to-one, and its inverse function is $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$.

If f maps $A \to B$ and A_1 is a subset of A, then we call the set of pairs $(a, f(a))$ with $a \in A_1$ the *restriction of f to A_1*, which is denoted by f/A_1 and which maps $A_1 \to B$. Clearly, if f is univalent, so is f/A_1.

EXAMPLE

Let $x \in \mathbf{R}$; then $f = x^2$ is a mapping $\mathbf{R} \to \mathbf{R}$ and contains the ordered pairs (a, b) with $a \in \mathbf{R}$, $b \in \mathbf{R}$ and $b = a^2$, for example, (1, 1), (2, 4), (0, 0), (−1, 1). This mapping is not from $\mathbf{R}$ onto $\mathbf{R}$ because $x^2 \geq 0$, so that $f(\mathbf{R})$ does not cover $\mathbf{R}$ but only $\mathbf{R}^+ \cup \{0\}$, where $\mathbf{R}^+$ is the set of real positive numbers. The function f is not univalent since we have $f(-a) = f(a)$, so there is no inverse function f^{-1}.

Let I be the interval $\{x|0 \leq x \leq 2\}$; then $f(I)$ is the interval $I_1 = \{y|0 \leq y \leq 4\}$, and $f^{-1}(I_1)$ is the interval $\{x|-2 \leq x \leq 2\}$. The restriction $f/\mathbf{R}^+ = x^2/\mathbf{R}^+$ has an inverse function, namely, the positive square root, and $f/\mathbf{R}^-$ also has an inverse function, namely, the negative square root.

EXAMPLE

Let z be a complex number; then $f = 1/z$ is a mapping which is defined throughout $\mathbf{C}\backslash\{0\}$. $f: \mathbf{C}\backslash\{0\} \to \mathbf{C}\backslash\{0\}$ is a mapping onto which is univalent since $1/z_1 = 1/z_2 \Rightarrow z_1 = z_2$; thus it is one-to-one. This mapping is called the *inversion mapping*; it contains the ordered pairs $(z, 1/z)$ with $z \in \mathbf{C}\backslash\{0\}$.

The inverse function f^{-1} exists since f is univalent and onto; f^{-1} contains the ordered pairs $(1/z, z)$ with $1/z \in \mathbf{C}\backslash\{0\}$. It is clear that $f^{-1} \circ f$ contains the pairs $(z, f^{-1}(f(z))) = (z, f^{-1}(1/z)) = (z, z)$ which defines the identity mapping. Set $z' = 1/z$; then f^{-1} contains the pairs $(z', 1/z')$, the same pairs as f. So f^{-1} is the same function as f.

Problems

0.1.: Show that equivalence in the sense of set theory is an equivalence relation.
0.2.: Show that the distributive law holds for the two operations $\cup$ and $\cap$:

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

and

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

---

4. *Sequences and countable sets*

DEFINITION A *sequence* is a set whose element are indexed by the positive integers:

$$a_1, a_2, a_3, \ldots, a_n, \ldots = \{a_n\}$$

EXAMPLE

$\{a_n = (-1)^n\}$ defines the sequence $-1, 1, -1, 1, \ldots$.

DEFINITION A set which is equivalent to the set of positive integers is said to be *countable.*

Clearly every countable set can be indexed in the form of a sequence; conversely, every sequence is countable since we have the one-to-one correspondence $n \leftrightarrow a_n$.

EXAMPLE

1 The set of all positive even integers is countable and forms the sequence

$$a_1 = 2, \quad a_2 = 4, \quad \ldots, \quad a_n = 2n, \quad \ldots$$

The set of all positive even integers is equivalent to the set of natural numbers. The one-to-one correspondence is easily seen to be given by:

$$\begin{array}{ccccccc} 1, & 2, & 3, & 4, & \ldots. & n, & \ldots \\ \updownarrow & \updownarrow & \updownarrow & \updownarrow & & \updownarrow & \\ 2, & 4, & 6, & 8, & \ldots. & 2n, & \ldots \end{array}$$

The function $f(n) = 2n$ which describes this one-to-one correspondence is univalent, since $2n = 2m \Leftrightarrow n = m$. Note that in this example a set is equivalent to one of its subsets. Clearly this is possible only for infinite sets.

2 The set of all integers is countable. Indeed, if we write the integers in the following order, $0, 1, -1, 2, -2, \ldots$, then $a_1 = 0$, $a_2 = 1$, $a_3 = -1$, $a_4 = 2, \ldots$, and in general $a_{2n} = n$ and $a_{2n+1} = -n$.

Let us establish two important properties of countable sets.

1. *Every infinite subset S' of a countable set S is countable.*

If $S = \{a_1, a_2, \ldots, a_n, \ldots\}$, then denote by a_1' the first element of S' you encounter in the sequence S, by a_2' the second, and so on. Then $S' = \{a_1', a_2', \ldots, a_n', \ldots\}$ is countable.

EXAMPLE

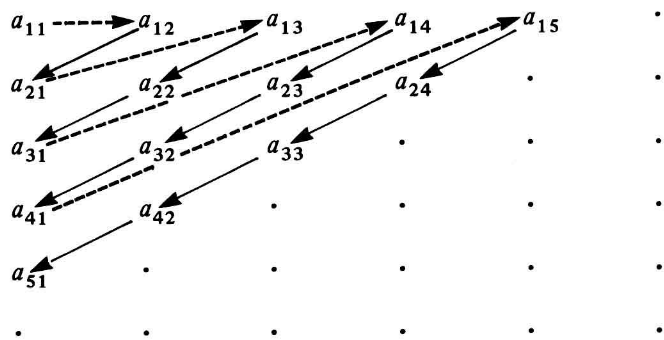
The odd integers are countable since they are an infinite subset of all the integers, which are countable.

2. *The union of a countable family of countable sets is countable.*

Let $\{S_1, S_2, \ldots, S_n, \ldots\}$ be the countable family of countable sets, and a_{ij} the jth element of S_i. All the elements a_{ij} may then be written in the following order:

$$\begin{matrix} a_{11}, a_{12}, a_{13}, \ldots \\ a_{21}, a_{22}, a_{23}, \ldots \\ a_{31}, a_{32}, a_{33}, \ldots \\ \ldots, \ldots, \ldots, \ldots \end{matrix}$$

We have listed the elements of S_1 in the first row, the elements of S_2 in the second row, and so on. Now we can write all these elements in the form of a sequence in the following way: $a_{11}, a_{12}, a_{21}, a_{13}, a_{22}, a_{31}, \ldots$, such that a_{mn} preceeds a_{rs} if $m + n < r + s$, or, in case $m + n = r + s$, if $m < r$. The order used is illustrated as follows:



This method of ordering elements a_{ij} into a sequence is called the diagonal method.

COROLLARY *The Cartesian products* **P** × **P** *and* **Z** × **Z** *are countable.*

EXAMPLE

The set of rational numbers is countable. In fact, any rational number is the ratio of two relatively prime integers p/q with $q > 0$. Hence, the set of rational numbers is equivalent to an infinite subset of **Z** × **Z**, where **Z** denotes the set of integers.

Problems

0.3.: Show that the set of real numbers is not countable.
0.4.: Show that the set of points of $\mathbf{R}^2$ with rational coordinates is countable. How could you generalize this result?
