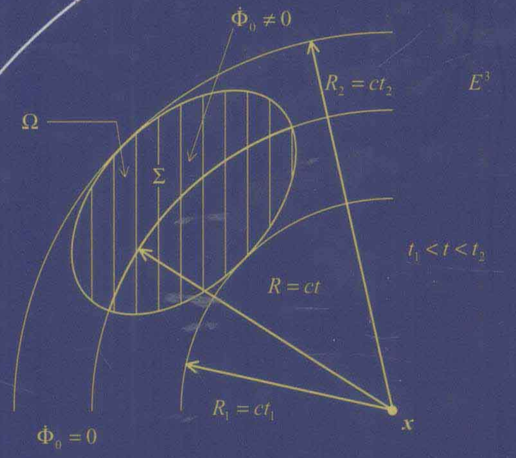
Mathematical Theory of **Elasticity**



Richard B. Hetnarski and Józef Ignaczak

MATHEMATICAL THEORY OF ELASTICITY

Richard B. Hetnarski

Department of Mechanical Engineering, Rochester Institute of Technology, Rochester, New York, U.S.A.

Józef Ignaczak

Center of Mechanics, Institute of Fundamental Technological Research Polish Academy of Sciences, Warsaw, Poland

Taylor & Francis

2004

Denise T. Schanck, *Vice President*
Robert L. Rogers, *Senior Editor*
Liliana Segura, *Editorial Assistant*
Savita Poornam, *Marketing Manager*
Randy Harinandan, *Marketing Assistant*

Dennis P. Teston, *Production Director*
Anthony Mancini Jr., *Production Manager*
Brandy Mui, *STM Production Editor*
Mark Lerner, *Art Director*
Daniel Sierra, *Cover Designer*

Published in 2004 by
Taylor & Francis
29 West 35th Street
New York, NY 10001

www.taylorandfrancis.com

Published in Great Britain by
Taylor & Francis
11 New Fetter Lane
London EC4P 4EE

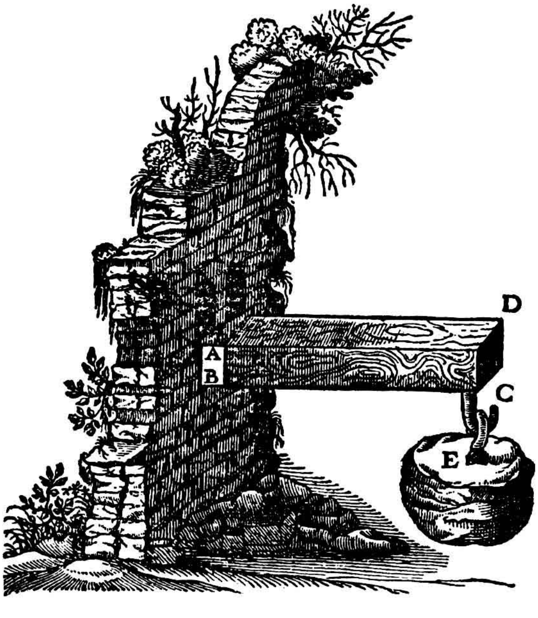
www.taylorandfrancis.co.uk

Printed in the United States of America on acid-free paper.

10 9 8 7 6 5 4 3 2 1

Library of Congress Cataloging-in-Publication Data

Hetnarski, Richard B.
Mathematical theory of elasticity / Richard B. Hetnarski, Józef Ignaczak.
p. cm.
Includes bibliographical references and index.
ISBN 1-59169-020-X
1. Elasticity. I. Ignaczak, Józef II. Title.
QA931.H57 2003
531′.382—dc21 2003046771


D
A
B
C
E

MATHEMATICAL THEORY OF ELASTICITY

We dedicate this book
to the memory
of our teacher
WITOLD NOWACKI.
He changed our lives.
The authors

Preface

The purpose of this book is to present the Mathematical Theory of Elasticity and its applications in a form suitable for a wide range of readers including graduate students, those preparing PhD theses, and those conducting research in continuum mechanics. Therefore, the book is not only a graduate textbook, but also serves as a complementary text to existing books on elasticity in that it provides classical results on elasticity as well as the new results obtained in recent years by various researchers, including the authors and their collaborators. Also, the book provides a bridge between the Mathematical Theory of Elasticity and Applied Elasticity through specific applications given in numerous Examples and Problems. It covers in one volume the areas of Elastostatics, Thermoelastostatics, Elastodynamics, and Thermoelastodynamics. Special emphasis is placed on the problems of elastodynamics and thermoelastodynamics, as most existing books on Elasticity deal mainly with elastostatics and thermoelastostatics.

The book consists of 13 chapters. The brief Chapter 1 tells about some of the creators of the Theory of Elasticity, Chapter 2 provides the Mathematical Preliminaries, Chapters 3–8 cover the Fundamentals of Linear Elasticity and applications, and Chapters 9–13 deal with applications only. Chapters 2 through 8 contain worked Examples that illustrate the theory involved, and at the end of each chapter, except Chapter 1, a number of Problems are included.

While making a selection of the material it was the authors' intention to provide the reader with both typical and new results of classical type, and to outline new areas of research. Therefore, Chapters 3–8 cover Kinematics, Motion and Equilibrium, Constitutive Relations, Formulation of Problems, and Variational Principles. New topics, such as the Convolutional Variational Principles of Elastodynamics due to M. E. Gurtin and the Pure Stress Formulations of Classical Elastodynamics that have emerged recently, are also discussed in detail. A new three-dimensional compatibility related variational principle of elastostatics corresponding to the two-dimensional result due to L. S. Leibenson [see, Theory of Elasticity, 2nd ed., Gostekhizdat, Moscow, 1947] is formulated in Chapter 5, while a number of unsolved PhD-level problems on incompatible elastodynamics are suggested at the end of Chapter 6. The Problems at the end of Chapter 7 on the complete solutions of elasticity include a Galerkin-type tensor solution as well as a Lamé-type tensor solution, both related to the pure stress treatment of elastodynamics. Chapters 9 and 10 treat, respectively, the solutions to particular three- and two-dimensional problems of elastostatics, and include detailed derivations of classical solutions such as Boussinesq's and Cerruti's solutions of three-dimensional elastostatics for a semispace subject to a concentrated boundary force, the solutions to two-dimensional counterparts of the Boussinesq's and Cerruti's problems, and the solutions to stationary three- and two-dimensional thermoelastic problems for a semispace involving thermal singularities and inclusions [see, Witold Nowacki, Thermoelasticity, Addison-Wesley Reading, Mass., 1962]. The Kirsch's problem for an infinite sheet with a circular hole subject to uniform tension at infinity (1898) is revisited in Chapter 10: what is new is the analysis of the associated displacements that is ignored in most of the books on elasticity. Also, the concept of a displacement concentration factor as opposed to a well-known stress concentration factor is presented.

Chapters 11 and 12 deal, respectively, with particular solutions of three- and two-dimensional problems of elastodynamics and thermoelastodynamics. Apart from the classical results, such as singular solutions of three- and two-dimensional elastodynamics and thermoelastodynamics for an infinite body, these chapters also include: (i) a tensorial classification of elastic waves, (ii) the solutions describing stress waves due to the initial stress and stress-rate fields in an infinite space, (iii) the dynamical thermal stresses produced by an instantaneous spherical temperature inclusion in an infinite body, and (iv) the Saint-Venant principle of elastodynamics in terms of stresses only. The authors obtained results in (ii)–(iv) only recently and these results are published here for the first time. Also, for the first time a closed-form solution that describes dynamical thermal stresses produced by an instantaneous source of heat in an infinite elastic sheet is published in Chapter 12. Finally, Chapter 13 covers a number of closed-form solutions to the one-dimensional initial boundary-value problems of isothermal and nonisothermal elastodynamics, including Green's functions for the infinite and semi-infinite solids as well as the solution that describes response of a semispace to short laser pulses.

Although the book is long, the authors had to make compromises, and a number of topics had to be omitted for lack of space. These topics, like, for example, complex variables method and numerical methods in solving differential equations and in performing inverse Laplace transformations, are treated well in easily accessible literature, and the authors feel they would contribute nothing new in presenting them.

Throughout the book the direct (that is, vectorial and tensorial) notation as well as the Cartesian coordinates are used. The associated terminology and general scheme of the notation follows that of M. E. Gurtin [The Linear Theory of Elasticity, Encyclopedia of Physics, Chief Editor S. Flügge, vol. VIa/2, Editor C. Truesdell, Springer, Berlin, 1972]. However, the authors have taken care to make the material easily accessible to the beginners and to those who have already gained an insight into the subject. The theory is developed in the style of Gurtin's treatise, but in addition to the presentation of basic concepts and theorems, Chapters 2–8 include 148 Examples that illustrate the theory and make the book more comprehensible. Specific applications taken up in the book are developed by using the integral representations of an external thermo-mechanical load to solve typical boundary-value problems of elastostatics and thermoelastostatics, and by taking advantage of the Laplace transform with respect to time when solving the initial boundary-value problems of elastodynamics and thermoelastodynamics.

One of the authors (JI) wishes to thank Dr. Charles Haines of Rochester Institute of Technology for allowing him to give a course on Theory of Elasticity to graduate students in the Department of Mechanical Engineering at RIT, during the winter quarter 1994–5. Notes prepared for the course as well as the material prepared by the authors over several years either working separately or jointly have developed into the book. In particular, the authors worked together during the first author's (RBH) numerous visits to the Institute of Fundamental Technological Research of the Polish Academy of Sciences in Warsaw, and the second author's (JI) visits at RIT as a Visiting Professor in 1994–5 and as a Short-Term Scholar in 1998.

The authors wish to thank Ms. Agata Buczyńska and Mrs. Grażyna Wasilewska for typing the manuscript in LaTeX, and Mrs. Mariola Rejmund for preparing the figures in an electronic form. The assistance of Mrs. Elżbieta

Mrówka-Matejewska in an over-all computer preparation of the book is also warmly acknowledged. Also, the authors express their sincere appreciation to the Taylor & Francis editors, especially to Bob Rogers, Brandy Mui, and Tony DeGeorge, for the effective and friendly cooperation during the publication of the book.

Richard B. Hetnarski
Józef Ignaczak
October 2003

Notation

Throughout the book a direct notation as well as Cartesian coordinates are used. Terminology and general scheme of the notation follow that of M. E. Gurtin [*The Linear Theory of Elasticity,* Encyclopedia of Physics, Chief Editor S. Flügge, vol. VIa/2, Editor C. Truesdell, Springer, Berlin, 1972]. In particular, scalars appear as italic light face letters, vectors are written as lower case letters in bold face, second-order tensors as upper case letters in bold face, and fourth-order tensors as upper case sans-serif letters in bold face. Also, within a section the same letters are used for quantities other than those listed below.

List of Symbols

Symbol	Name
$\mathbf{A}$	Second-order tensor, Beltrami solution, thermal expansion tensor
$\mathbf{A}(\mathbf{m})$	Acoustic tensor for a direction $\mathbf{m}$
B	Body
$\mathbf{B}$	Second-order tensor
$\boldsymbol{\mathsf{C}}$	Elasticity tensor
D	Torsional rigidity
$\mathbf{D}$	Finite strain tensor
E	Young's modulus
E^3	Three-dimensional Euclidean space
E^2	Two-dimensional Euclidean space
$\mathbf{E}$	Infinitesimal strain tensor
$\mathbf{E}^{\perp}$	Normal part of $\mathbf{E}$ with respect to a plane
$\mathbf{E}^{\parallel}$	Tangential part of $\mathbf{E}$ with respect to a plane

Symbol	Name
$\mathcal{E}_E(\mathbf{E})$	Strain energy density of a progressive wave
$\mathcal{E}_E(\mathbf{E}^\perp)$	Normal strain energy density
$\mathcal{E}_S(\mathbf{S})$	Stress energy density of a progressive wave
$\mathcal{E}_S(\mathbf{S}^\perp)$	Normal stress energy density
$\mathcal{E}_S(\mathbf{S}^\parallel)$	Tangential stress energy density
F	Airy stress function, force
$\mathbf{F}$	Deformation gradient
$F\{\cdot\}$	Functional
G	Shear modulus, Green's function
$H(\cdot)$	Heaviside's function
$\mathbf{H}$	Harmonic second-order tensor field
H	Compatibility related fourth-order tensor
I	Moment of inertia of a cross section
J	Polar moment of inertia of a cross section
K	Stress concentration factor
K_r	Displacement concentration factor
$K(t)$	Kinetic energy
K	Compliance tensor
L	Laplace transform, length
M	Bending moment
M_3	Torsion moment
$\mathbf{M}$	Stress-temperature tensor
$\mathcal{M}_{\mathrm{x},ct}(f)$	Mean value of a function f over the surface of a sphere with its center at $\mathbf{x}$ and of radius ct
$\mathcal{N}_{\mathrm{x},ct}(f)$	Two-dimensional counterpart of $\mathcal{M}_{\mathrm{x},ct}(f)$
$\mathbf{O}$	Origin, zero vector, zero tensor
P	Part of B, concentrated force
$P(t)$	Stress power
Q	Heat supply field, shear force
$\mathbf{Q}$	Orthogonal tensor
R	Region in E^3, distance between two points
$\mathbf{S}$	Stress tensor
$\widehat{\mathbf{S}}(B)$	Mean stress
$\mathbf{S}^\perp$	Normal part of $\mathbf{S}$ with respect to a plane
$\mathbf{S}^\parallel$	Tangential part of $\mathbf{S}$ with respect to a plane
T	Temperature change, time interval
$U_C\{\mathbf{E}\}$	Strain energy
$\mathcal{U}(t)$	Total energy of B at time t

Symbol	Name
$U(l,t)$	Stress energy of a semi-infinite cylinder $B(l)$ stored over time interval $[0,\ t]$
$\mathcal{V}$	Vector space associated with E^3
$\mathbf{W}$	Rotation tensor
W	Internal heat generated per unit of volume per unit of time
$W(\mathbf{E})$	Stored energy function
$W'(\mathbf{S})$	Complementary strain energy
$\widehat{W}(\mathbf{S})$	Stress energy density
$W_t\{\cdot\}$	Functional involving convolutions
$\mathbf{a}$	Direction of motion of a progressive wave
$\mathbf{b}$	Body force
c	Velocity of propagation, specific heat
c_1	Irrotational velocity
c_2	Isochoric velocity
$c(\mathbf{S}^0)$	Velocity of a stress wave
$\mathbf{e}$	Unit vector along the axis of symmetry of a transversely isotropic body
$\mathbf{e}_i$	Orthonormal basis
$\mathbf{f}$	Pseudo-body force field
$\mathbf{g}$	Galerkin vector field
$\mathbf{g}(P)$	Linear momentum of P
$\mathbf{h}(P)$	Angular momentum of P
i	$\sqrt{-1}$, function with the values $i(t)=t$
k	Bulk modulus, polar radius of gyration of a cross section, spring stiffness, thermal conductivity
$\mathbf{k}$	Unit vector along x_3 axis
ℓ	Concentrated force
$\mathbf{m}$	Direction of propagation
m	The constant $\frac{1+\nu}{1-\nu}\alpha$, also mass of P
$\mathbf{n}$	Outward unit normal on ∂B
p	Pressure, admissible process, elastic process, thermoelastic process
$\mathbf{s}$	Surface traction
$\widehat{\mathbf{s}}$	Prescribed surface traction
s	Admissible state, elastic state, thermoelastic state
t	Time
$\mathbf{u}$	Displacement vector

Symbol	Name
$\widehat{\mathbf{u}}$	Prescribed displacement on boundary
$\mathbf{u}_0$	Initial displacement
$\dot{\mathbf{u}}_0$	Initial velocity
$v(B)$	Volume of B
$\mathbf{w}$	Rigid displacement
$\mathbf{x}$, $\mathbf{y}$	Points in space (vectors)
x_i	Cartesian components of $\mathbf{x}$
α	Coefficient of linear thermal expansion, angle of twist, scalar field in a tensorial solution of elastodynamics
β	Vector field in a tensorial solution of elastodynamics
γ	The constant $(3\lambda + 2\mu)\alpha$
$\delta(\cdot)$	Dirac delta function
δ_{ij}	Kronecker's delta
$\delta V(B)$	Volume change
ϵ_{ijk}	Three-dimensional alternator
$\epsilon_{\alpha\beta}$	Two-dimensional alternator
θ	Absolute temperature
θ_0	Reference temperature
κ	Thermal diffusivity
λ	Lamé constant, wave length
μ	Lamé constant (shear modulus)
ν	Poisson's ratio
ρ	Density
σ	Normal component of a stress vector
τ	Dimensionless time, shear stress
φ	Scalar field in Boussinesq-Papkovitch-Neuber solution, scalar field in Green-Lamé solution
ϕ	Thermoelastic displacement potential, Prandtl's stress function, biharmonic function
χ	Biharmonic scalar field in Love's solution
$\boldsymbol{\chi}$	Second-order tensor field of Galerkin type in elastodynamics
ψ	Warping function
$\boldsymbol{\psi}$	Vector field in Boussinesq-Papkovitch-Neuber solution, vector field in Green-Lamé solution

Symbol	Name
$\boldsymbol{\omega}$	Rotation vector, vector field in a tensorial solution of elastodynamics
$\mathbf{1}$	Unit tensor
sym	Symmetric part of a tensor
skw	Skew part of a tensor
tr	Trace of a tensor
$\otimes$	Tensor product of two vectors
∇	Gradient
$\widehat{\nabla}$	Symmetric gradient
curl	Curl
div	Divergence
$\Delta = \nabla^2$	Laplacian
$\Box_1^2, \Box_2^2$	Wave operators
$*$	Convolution
$[\![\cdot]\!]$	Jump in a function
da	Element of area
dv	Element of volume
$\dot{(\ )}$	Time derivative
$(\ )^T$	Transpose of a tensor

Some Quantities in SI Units

$[I] = [J]$ (area moments of inertia)	m^4
$[S_{ij}] = [G] = [E] = [k] = [s_i] = [p] = [\lambda] = [\mu]$	$Pa = N/m^2$
(E = Young's modulus, k = bulk modulus, p = pressure, λ, μ = Lamé constants)	
$[Q] = [l_i] = [F]$ (Q = shear force, F = force)	N
$[Q]$ (heat supply field)	K/s
$[Q_o]$ (in Section 11.2)	$K{\cdot}m^3$
$[Q_o]$ (in Section 12.2)	$K{\cdot}m^2$
$[Q_o]$ (in Section 13.2)	K/s
$[S_o]$ [in Eqs. (11.2.24) and (13.2.10)]	$kg/(m{\cdot}s^2)$ or N/m^2
$[T] = [\theta] = [\theta_o]$	K
$[W]$ (stored energy)	J/m^3
$[W]$ (internal heat generated per unit of volume per unit of time)	$J/(m^3{\cdot}\ s) = W/m^3$
$[b_i]$ (body force per unit of volume)	N/m^3
$[c] = [c_1] = [c_2]$ (velocity)	m/s
$[c]$ (specific heat)	J/(kg·K)
$[g(P)_i]$ (linear momentum)	(kg·m)/s
$[h(P))_i]$ (angular momentum)	$(kg{\cdot}m^2)/s$ or N·m·s
$[k]$ (thermal conductivity)	W/(m·K)
$[m] = [\alpha]$	1/K
$[u_i] = [(u_o)_i] = [w_i] = [L]$	m
$[\lambda = (3\lambda + 2\mu)\alpha]$	$N/(m^2{\cdot}\ K)$
$[\kappa]$ (thermal diffusivity), $\kappa = k/(\rho \cdot c)$	m^2/s
$[\rho]$ (density)	kg/m^3
$[\phi]$ (thermoelastic displacement potential)	m^2

Values of the specific heat c, the thermal conductivity k, the density ρ, and the thermal diffusivity κ for common materials may be found in M. N. Özişik, *Heat Conduction,* John Wiley, New York, 1980, p. 2–7.

Values of the coefficient of linear thermal expansion α and Young's modulus E for common materials may be found in N. Noda, R. B. Hetnarski, and Y. Tanigawa, *Thermal Stresses,* first edition, Lastran, Rochester, 2000, p. 7, or second edition, Taylor & Francis, New York, 2003, p. 7.