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Shigeru Iitaka

Algebraic Geometry

An Introduction to Birational Geometry of Algebraic Varieties

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Preface

The aim of this book is to introduce the reader to the geometric theory of algebraic varieties, in particular to the birational geometry of algebraic varieties.

This volume grew out of the author's book in Japanese published in 3 volumes by Iwanami, Tokyo, in 1977. While writing this English version, the author has tried to rearrange and rewrite the original material so that even beginners can read it easily without referring to other books, such as textbooks on commutative algebra. The reader is only expected to know the definition of Noetherin rings and the statement of the Hilbert basis theorem.

The new chapters 1, 2, and 10 have been expanded. In particular, the exposition of D-dimension theory, although shorter, is more complete than in the old version. However, to keep the book of manageable size, the latter parts of Chapters 6, 9, and 11 have been removed.

I thank Mr. A. Sevenster for encouraging me to write this new version, and Professors K. K. Kubota in Kentucky and P. M. H. Wilson in Cambridge for their careful and critical reading of the English manuscripts and typescripts. I held seminars based on the material in this book at The University of Tokyo, where a large number of valuable comments and suggestions were given by students Iwamiya, Kawamata, Norimatsu, Tobita, Tsushima, Maeda, Sakamoto, Tsunoda, Chou, Fujiwara, Suzuki, and Matsuda.

Fall 1981 Shigeru Iitaka

Contents

Introduction

The purpose of algebraic geometry is to study comprehensively varieties defined by a set of polynomial equations in many variables

$$f_1(X_1, \ldots, X_n) = \cdots = f_r(X_1, \ldots, X_n) = 0.$$

Properties of varieties should be independent of the choice of coordinate systems. For example, the variety defined by $X_2 = 0$ ($r = 1$, $n = 2$) is equivalent to that defined by $Y_1 - Y_2^2 = 0$ ($r = 1$, $n = 2$) under the invertible transformation $X_1 = Y_2$, $X_2 = Y_1 - Y_2^2$. This equivalence is interpreted as the existence of an isomorphism of rings

$$\frac{k[X_1, X_2]}{(X_2)} \cong \frac{k[Y_1, Y_2]}{(Y_1 - Y_2^2)}.$$

Thus, the study of a set of polynomial equations can be reduced to the study of a commutative ring $k[X_1, \ldots, X_n]/\mathfrak{a}$, where $\mathfrak{a}$ is an ideal generated by $f_1, \ldots, f_r$. From this viewpoint, one arrives naturally at the concepts of affine schemes and then of schemes.

However, ever since the last century, it has been believed that the more essential properties of varieties are those which are birationally invariant.

A plane curve is defined by an irreducible polynomial $\varphi(X, Y)$. The degree of φ is said to be the degree of the curve. Plane curves defined by irreducible polynomials f_1 and f_2 are birationally equivalent if the field $Q(k[X, Y]/(f_1))$ is isomorphic to $Q(k[X, Y]/(f_2))$, where $Q(R)$ denotes the field of fractions of the integral domain R.

The degree is not, however, a birational invariant. As Abel noted, the number of linearly independent Abelian differentials of the first kind (also called regular 1-forms) on a given curve C is more important than the degree, since it is a birational invariant. This number is called the genus of

C, denoted by $g(C)$. Curves can be classified into the following three classes according to their genera:

The class I: $g(C) = 0$.

The class II: $g(C) = 1$.

The class III: $g(C) \geq 2$.

A similar birational classification for 2-dimensional varieties (called surfaces) was obtained by Italian algebraic geometers around the beginning of our own century.

Given a variety V of dimension n, many birational invariants can be defined, such as the plurigenera, the i-th irregularity, and the Kodaira dimension. Let $\kappa(V)$ denote the Kodaira dimension of V, which can take the values $-\infty, 0, 1, \ldots, n$. By means of the Kodaira dimension, varieties of dimension n can be classified into $n + 2$ classes. When $n = 1$, this classification agrees with that given by the genus.

Many fundamental properties of the Kodaira dimension have been found, giving some basic information about the structure of varieties.

Let V be a variety of dimension n and suppose that $\kappa(V) \geq 0$. Then by Theorem 10.3 (fibering theorem), there exists a dominating morphism $f: V^* \to W$ such that (1) V^* is birationally equivalent to V, (2) $\dim W = \kappa(V)$, (3) general fibers $f^{-1}(x)$ are irreducible, and (4) for a (strictly) general point x of W, $\kappa(f^{-1}(x)) = 0$.

Varieties V with $\kappa(V) = n$ are said to be of general type or of hyperbolic type. Roughly speaking, almost all varieties are of hyperbolic type and these have rather general properties in common. For example, if V is of hyperbolic type, then the automorphism group $\mathrm{Aut}(V)$ of V is a finite group.

The number of linearly independent regular 1-forms on a complete nonsingular variety V is also a birational invariant, denoted by $q(V)$. In particular, if $\dim V = 1$, then $q(V)$ turns out to be the genus of V. In general, an Abelian variety of dimension $q(V)$, the Albanese variety $\mathrm{Alb}(V)$, is associated with V, together with the Albanese map $\alpha_V: V \to \mathrm{Alb}(V)$.

Recently Kawamata proved that if $\kappa(V) = 0$, then α_V is surjective and general fibers $\alpha_V^{-1}(x)$ are irreducible. Thus in the case where $\kappa(V) = 0$ and $q(V) > 0$, the structure of V can be studied using Albanese maps. However, nothing is known about V when $\kappa(V) = q(V) = 0$. If $\dim V = 2$, it has been shown that such a V is birationally equivalent to a $K3$ surface or an Enriques surface.

In the case where $\kappa(V) = -\infty$ and $q(V) > 0$, consider again the Albanese map $\alpha_V: V \to \mathrm{Alb}(V)$. One has a morphism $\psi: V \to Z$ obtained from the Stein factorization of $\alpha_V: V \to \alpha_V(V)$. Then it is conjectured that $\kappa(\psi^{-1}(x)) = -\infty$ for a general point x of Z. Actually, this has been proved for $n \leq 3$ (by Enriques for $n = 2$; by Viehweg for $n = 3$). The case where $\kappa(V) = -\infty$ and $q(V) = 0$ seems the most difficult case to study. When $n = 2$, such a V is a rational surface, i.e., a surface birationally equivalent to $\mathbf{P}^1 \times \mathbf{P}^1$. This fact, discovered by Castelnuovo, was the starting point of the classification theory of algebraic surfaces by the Italian school. But in the higher dimensional case, nothing is known about such V.

Chapter 10 may serve as a guide to this rapidly developing theory of birational classification of varieties.

It is unreasonable to say that only birationally invariant properties are worth studying. For instance, the affine line $\mathbf{A}^1$ is quite different from $G_m = \mathbf{A} - \{0\}$, which are both very important. However, they are birationally equivalent.

Any variety is birationally equivalent to a complete variety. Thus, when considering noncomplete varieties and studying their properties, we can no longer use birational equivalence. However, in this case, a more delicate equivalence relation, called proper birational equivalence, is introduced (see Chapter 2). One can find many proper-birational invariants such as the logarithmic genera, logarithmic irregularities, and logarithmic Kodaira dimension, which are defined by making use of logarithmic forms. A proper birational equivalence between affine normal varieties is just an isomorphism between them; hence the corresponding normal rings are isomorphic. Thus, in our study of proper birational properties of varieties, the theory of (normal) rings and birational geometry are unified; thus the theorems on Kodaira dimension could be translated into ring theory and so on.

Algebraic geometry should be a synthesis of algebra and geometry. But, in practice, it has been an algebraic approach to geometry. Our new birational geometry (e.g., proper birational geometry) is not only a revival of old birational geometry but is also a beginning of some grand unified theory of algebra and geometry.

Chapter 1

Schemes

§1.1 Spectra of Rings

a. We begin by defining spectra of commutative rings with identity, which are the base spaces of the affine schemes introduced in §1.11.

In all that follows, commutative rings A, B, ... with identity elements 1_A, 1_B, ... are referred to simply as *rings*, and ring homomorphisms $\varphi: A \to B$ are assumed to satisfy $\varphi(1_A) = 1_B$.

Definition. The *spectrum* of a ring A is the set of all prime ideals of A, denoted by Spec A.

Note that the ring A itself is not considered to be a prime ideal.

If $\varphi : A \to B$ is a ring homomorphism and $\mathfrak{p}$ is a prime ideal of B, then $\varphi^{-1}(\mathfrak{p})$ is also a prime ideal of A. We note that $1_A \notin \varphi^{-1}(\mathfrak{p})$, since $1_B \notin \mathfrak{p}$ and $\varphi(1_A) = 1_B$.

Definition. The mapping ${}^a\varphi$: Spec $B \to$ Spec A defined by ${}^a\varphi(\mathfrak{p}) = \varphi^{-1}(\mathfrak{p})$ is said to be the *mapping associated with* φ.

EXAMPLE 1.1. (i) For the trivial ring 0, we have Spec $0 = \varnothing$.

(ii) If k is a field (a field is always assumed to be nontrivial, i.e., $k \neq \{0\}$), then Spec $k = \{(0)\}$.

(iii) Spec $\mathbb{Z} = \{(0)\} \bigcup \{(p) \mid p \text{ is a prime number}\}$.

(iv) If $k[\mathrm{X}]$ is a ring of polynomials over an algebraically closed field k, then

$$\operatorname{Spec} k[\mathrm{X}] = \{(0)\} \bigcup \{(\mathrm{X} - \alpha) \mid \alpha \in k\},$$

which can be written as $\operatorname{Spec} k[\mathsf{X}] = \{*\} \bigcup k$ with the abbreviations $* = (0)$, $\alpha = (\mathsf{X} - \alpha)$.

(v) If $k[\mathsf{X}, \mathsf{Y}]$ is a polynomial ring in two variables over an algebraically closed field k, then

$$\operatorname{Spec} k[\mathsf{X}, \mathsf{Y}] = \{(0)\} \bigcup \{(f) \mid f \text{ is a nonconstant irreducible polynomial in } \mathsf{X} \text{ and } \mathsf{Y}\} \bigcup \{(\mathsf{X} - \alpha, \mathsf{Y} - \beta) \mid (\alpha, \beta) \in k^2\}.$$

PROOF OF (v). Clearly, it suffices to show that every nonprincipal prime ideal $\mathfrak{p}$ is of the form $(\mathsf{X} - \alpha, \mathsf{Y} - \beta)$. Let $f \in \mathfrak{p}\backslash\{0\}$ be a polynomial with $\deg_{\mathsf{Y}} f$ minimal such that f is irreducible. Since $\mathfrak{p}$ is nonprincipal, there is an element $g \in \mathfrak{p}\backslash(f)$. By the Euclidean algorithm in the ring $k(\mathsf{X})[\mathsf{Y}]$, there is a $p \in k[\mathsf{X}]\backslash(0)$ and $q, r \in k[\mathsf{X}, \mathsf{Y}]$ satisfying $pg = qf + r$, where either $r = 0$ or $\deg_{\mathsf{Y}} r < \deg_{\mathsf{Y}} f$. It follows that $r = 0$ by the choice of f and because $r = pg - qf \in \mathfrak{p}$. But then $p \in (f)$ because (f) is a prime ideal, $pg = qf \in (f)$, and $g \notin (f)$. Thus $\deg_{\mathsf{Y}} f \leq \deg_{\mathsf{Y}} p = 0$; i.e., $f \in k[\mathsf{X}]$. Since k is algebraically closed and $f \in k[\mathsf{X}]$ is irreducible, f must be linear and so $\mathfrak{p}$ contains a polynomial of the form $\mathsf{X} - \alpha$ with $\alpha \in k$. Interchanging the roles of X and Y, one also sees that $\mathfrak{p}$ contains a polynomial of the form $\mathsf{Y} - \beta$ with $\beta \in k$. But then $\mathfrak{p}$ contains the maximal ideal $(\mathsf{X} - \alpha, \mathsf{Y} - \beta)$ and hence must be equal to it (cf. Exercise 1.1).

The reader can easily verify that the maximal ideals of $k[\mathsf{X}, \mathsf{Y}]$ are precisely those of the form $(\mathsf{X} - \alpha, \mathsf{Y} - \beta)$ with $(\alpha, \beta) \in k^2$. □

b. We shall introduce a topology on Spec A. For any ideal $\mathfrak{a}$ of A, define the set $V(\mathfrak{a})$ to be $\{\mathfrak{p} \in \operatorname{Spec} A \mid \mathfrak{p} \supseteq \mathfrak{a}\}$, and for any $f \in A$, define $V(f)$ to be $\{\mathfrak{p} \in \operatorname{Spec} A \mid \mathfrak{p} \ni f\}$. Then, $V(f) = V(fA)$ and the following properties are easily verified:

(i) $V(0) = \operatorname{Spec} A$, $V(1) = \varnothing$.
(ii) If $\mathfrak{a}$ and $\mathfrak{b}$ are ideals such that $\mathfrak{a} \subseteq \mathfrak{b}$, then $V(\mathfrak{a}) \supseteq V(\mathfrak{b})$.
(iii) $V(\mathfrak{a} \cap \mathfrak{b}) = V(\mathfrak{a}\mathfrak{b}) = V(\mathfrak{a}) \cup V(\mathfrak{b})$. In particular, $V(fg) = V(f) \cup V(g)$ for all $f, g \in A$.
(iv) If $\{\mathfrak{a}_\lambda \mid \lambda \in \Lambda\}$ is a set of ideals of A, then $V(\sum_{\lambda \in \Lambda} \mathfrak{a}_\lambda) = \bigcap_{\lambda \in \Lambda} V(\mathfrak{a}_\lambda)$.
(v) For any ideal $\mathfrak{a}$ of A, if $\varphi\colon A \to A/\mathfrak{a}$ is the natural homomorphism then ${}^a\varphi\colon \operatorname{Spec}(A/\mathfrak{a}) \to \operatorname{Spec} A$ is one-to-one and $\operatorname{Im} {}^a\varphi = V(\mathfrak{a})$.

Definition. The topology on Spec A is introduced by taking the sets in $\{V(\mathfrak{a}) \mid \mathfrak{a} \text{ ideals of } A\}$ as the closed sets.

The open sets of this topology are just those of the form

$$D(\mathfrak{a}) = \operatorname{Spec} A \backslash V(\mathfrak{a}) = \{\mathfrak{p} \in \operatorname{Spec} A \mid \mathfrak{p} \not\supseteq \mathfrak{a}\}.$$

When expressed in terms of the $D(\mathfrak{a})$, properties (i) through (v) will be referred to as properties (i′) through (v′), respectively. For example, in view of the property (iv′), if $\mathfrak{a}$ is the ideal generated by $\{f_\lambda \mid \lambda \in \Lambda\}$, then

$$D(\mathfrak{a}) = \bigcup_{\lambda \in \Lambda} D(f_\lambda),$$

where $D(f) = \operatorname{Spec} A \backslash V(f) = \{\mathfrak{p} \in \operatorname{Spec} A \mid \mathfrak{p} \not\ni f\}$.

Note that the sets $D(f)$ form an open base for the topology of Spec A.

The following properties of the mapping ${}^a\varphi$: Spec $B \to$ Spec A associated with $\varphi: A \to B$ are easily verified.

Proposition 1.1

(i) ${}^a\varphi$ *is continuous. More precisely, if* $f \in A$, *then* $({}^a\varphi)^{-1}(D(f)) = D(\varphi(f))$, *and if* $\mathfrak{a}$ *is an ideal of* A, *then* $({}^a\varphi)^{-1}(V(\mathfrak{a})) = V(\mathfrak{a}B)$, *where* $\mathfrak{a}B$ *is the ideal of* B *generated by* $\varphi(\mathfrak{a})$.

(ii) *For any ideal* $\mathfrak{b}$ *of* B, *one has*

$${}^a\varphi(V(\mathfrak{b})) \subseteq V(\varphi^{-1}(\mathfrak{b})).$$

(iii) *If* $\varphi: A \to A/\mathfrak{a}$ *is the natural homomorphism, then* ${}^a\varphi$: Spec $A/\mathfrak{a} \to$ Spec A *is a homeomorphism of* Spec $A/\mathfrak{a}$ *onto* $V(\mathfrak{a})$.

(iv) *If* $\varphi: A \to B$ *is surjective, then* ${}^a\varphi$ *is a homeomorphism of* Spec B *onto the closed subset* $V(\operatorname{Ker} \varphi)$ *of* Spec A.

(v) *On the other hand, if* ${}^a\varphi$: Spec $B \to$ Spec A *is surjective, then* $V(\operatorname{Ker} \varphi) = V(\varphi^{-1}(0)) = \operatorname{Spec} A$. *Hence* Ker φ *is a subset of every prime ideal of* A.

For the proof of the next propositions, we need a result from ring theory.

Definition. If $\mathfrak{a}$ is an ideal of A, the *radical* $\sqrt{\mathfrak{a}}$ is $\{a \in A \mid a^m \in \mathfrak{a}$ for some integer $m > 0\}$. $\sqrt{(0_A)}$ is said to be the *nilradical* of A, which consists of all nilpotent elements of A.

$\sqrt{\mathfrak{a}}$ is an ideal containing $\mathfrak{a}$.

c. The following result is a key lemma in the first stage of the theory of spectra.

Lemma 1.1. *If* $\mathfrak{a}$ *is a proper ideal (i.e.,* $\mathfrak{a} \neq A$*) of* A, *then there exists a maximal ideal containing* $\mathfrak{a}$.

Proof. Let $\mathfrak{F}$ be the set of all proper ideals of A containing $\mathfrak{a}$. Then $\mathfrak{F}$ is not empty, since $\mathfrak{a} \in \mathfrak{F}$. The set $\mathfrak{F}$ is naturally ordered by set inclusion, i.e., $\mathfrak{a}_\lambda \le \mathfrak{a}_\mu$ if and only if $\mathfrak{a}_\lambda \subseteq \mathfrak{a}_\mu$. We shall show that $\mathfrak{F}$ is an inductively ordered set. In fact, letting $\{\mathfrak{a}_\lambda \mid \lambda \in \Lambda\}$ be an arbitrary linearly ordered subset of $\mathfrak{F}$, define $\mathfrak{a}_* = \bigcup_{\lambda \in \Lambda} \mathfrak{a}_\lambda$, which becomes a proper ideal, i.e., $\mathfrak{a}_* \in \mathfrak{F}$ and $\mathfrak{a}_\lambda \le \mathfrak{a}_*$ for any $\lambda \in \Lambda$. Hence, the ordered set $\mathfrak{F}$ is inductive.

By Zorn's lemma, $\mathfrak{F}$ has a maximal element $\mathfrak{m}$, which is a maximal ideal containing $\mathfrak{a}$. □

Corollary

(i) Spec $A = \emptyset$ *if and only if* $A = 0$.
(ii) $V(\mathfrak{a}) = \emptyset$ *if and only if* $\mathfrak{a} = A$.

EXAMPLE 1.2. Let $a_1, \ldots, a_r \in A$. If there is no prime ideal containing $a_1, \ldots, a_r$, then there exist $b_1, \ldots, b_r$ in A such that $a_1 b_1 + \cdots + a_r b_r = 1$.

PROOF. Let $\mathfrak{a} = \sum_{j=1}^{r} a_j A$. If $\mathfrak{a} \neq A$, then there exists a prime ideal $\mathfrak{p}$ containing $\mathfrak{a}$ by Lemma 1.1. □

§1.2 Examples of Spectra as Topological Spaces

a. In $X = \operatorname{Spec} \mathbb{Z}$, the closed sets are X, $\emptyset$, and the sets of the form $\{(p_1), \ldots, (p_s)\}$, i.e., the finite sets of prime numbers.

b. Let k be an algebraically closed field and $k[\mathsf{X}]$ be the corresponding polynomial ring. Then as was seen in Example 1.1.(iv), Spec $k[\mathsf{X}]$ can be written as $\{*\} \cup k$. Since any ideal of $k[\mathsf{X}]$ is either (0), (1), or (f), where $f \in k[\mathsf{X}] \backslash k$, the closed sets are $\{*\} \cup k$, $\emptyset$, and the sets of the form $\{\lambda_1, \ldots, \lambda_s\}$ where the λ_i are the roots of $f(x) = 0$ for such an f.

Note that in this case a union of finitely many closed sets $F_1, F_2, \ldots, F_r$ none of which is the whole space is not the whole space. In other words, if $U_1, \ldots, U_r$ are nonempty open sets, then $U_1 \cap \cdots \cap U_r$ is not empty.

Now, let $A = k[\mathsf{X}, \mathsf{Y}]$ as in Example 1.1.(v). Then

$$\operatorname{Spec} A = \{*\} \cup \{(f) \mid f \in k[\mathsf{X}, \mathsf{Y}] \backslash k \text{ is irreducible}\} \cup k^2.$$

If we consider k^2 as the topological space with the topology induced from Spec A, then the closed sets are k^2, $\emptyset$, and finite unions of the finite sets and the sets of the form $\{(a, b) \in k^2 \mid \varphi(a, b) = 0\}$ for $\varphi \in k[\mathsf{X}, \mathsf{Y}]$. This is easily checked, since A is Noetherian.

c. Let $A = k[[\mathsf{X}]]$, the formal power series ring over a field k. Then Spec $A = \{(0), (\mathsf{X})\}$. The closed sets are $\{(0), (\mathsf{X})\}$, $\{(\mathsf{X})\}$, and $\emptyset$. Hence, the closure of (0) is the whole space. Spec A is a topological space with two points which is not discrete.

§1.3 Rings of Fractions, the Case A_f

Let A be a ring. For any element f of A, define A_f to be $A[\mathsf{X}]/(f\mathsf{X} - 1)$, where $A[\mathsf{X}]$ is the ring of polynomials over A. Letting $\psi_f(a) = a \bmod (f\mathsf{X} - 1)$ for $a \in A$, and $\xi = \mathsf{X} \bmod (f\mathsf{X} - 1)$, one has the ring homomorphism

$\psi_f\colon A \to A_f$ and ξ satisfies $\psi_f(f)\cdot\xi = 1$. Thus we denote ξ by $1/f$. A_f is then generated by $1/f$ as an A-algebra, i.e., $A_f = A[1/f]$. For simplicity, we write $a/1$ instead of $\psi_f(a)$.

Proposition 1.2.

(i) $a/1 = b/1$ *if and only if* $f^n a = f^n b$ *for some* $n \geq 0$.

(ii) $A_f = 0$ *if and only if* f *is a nilpotent element.*

(iii) (Universal mapping property). *For every ring homomorphism* $\varphi\colon A \to B$ *such that* $\varphi(f)$ *has a multiplicative inverse, there is a unique ring homomorphism* $\varphi^{\#}\colon A_f \to B$ *such that* $\varphi^{\#} \circ \psi_f = \varphi$. (*This means that among all pairs* (B, φ) *of rings* B *and homomorphisms* $\varphi\colon A \to B$ *such that* $\varphi(f)$ *is invertible, the pair* (A_f, ψ_f) *is the universal one.*)

(iv) ${}^a\psi_f\colon \operatorname{Spec} A_f \to \operatorname{Spec} A$ *is a homeomorphism onto* $D(f)$, *i.e.,* $\operatorname{Spec} A_f \approx D(f)$.

Proof. (i) It suffices to prove this for $b = 0$. $a/1 = 0$ if and only if $a \in (f\mathsf{X} - 1)$, i.e., there exist $n \geq 0$ and $b_0, \ldots, b_n \in A$ such that $a = (f\mathsf{X} - 1)(b_0 + b_1\mathsf{X} + \cdots + b_n\mathsf{X}^n)$, i.e.,

$$a = -b_0, fb_0 - b_1 = 0, \ldots, fb_{n-1} - b_n = 0, fb_n = 0. \qquad (*)$$

If $(*)$ holds, then $f^{n+1}a = -f^{n+1}b_0 = \cdots = fb_n = 0$. Conversely, if $f^{n+1}a = 0$, by taking $b_i = -f^i a$ for $0 \leq i \leq n$, then $(*)$ holds. So there exists $n \geq 0$ such that $f^{n+1}a = 0$ if and only if $(*)$ holds, which is equivalent to $a/1 = 0$.

(ii) This follows from (i) and the fact that $A_f = 0$ if and only if $1/1 = 1_{A_f} = 0$.

(iii) Define a ring homomorphism $\Phi\colon A[\mathsf{X}] \to B$ by $\Phi|_A = \varphi$ and $\Phi(\mathsf{X}) = 1/\varphi(f)$. Then $\Phi(f\mathsf{X} - 1) = \Phi(f)\Phi(\mathsf{X}) - 1 = \varphi(f)\cdot(1/\varphi(f)) - 1 = 0$; hence Φ determines a ring homomorphism $\varphi^{\#}\colon A_f \to B$ such that $\varphi^{\#}(a/f^r) = \varphi(a)/\varphi(f)^r$, i.e., $\varphi^{\#} \circ \psi_f = \varphi$. Since $A_f = A[1/f]$, any ring homomorphism $\varphi'\colon A_f \to B$ such that $\varphi' \circ \psi_f = \varphi$ becomes $\varphi^{\#}$.

(iv) The proof is left to the reader, since the proof of the more general case will be given in §1.4 (cf. Lemma 1.3). □

Corollary. *The kernel of* ψ_f *is the ideal* $I(f) = \{a \in A \mid af^m = 0$ *for some* $m \geq 1\}$.

Proposition 1.3. *Let* $\mathfrak{a}$ *be a proper ideal of a ring* A. *Then*

$$\sqrt{\mathfrak{a}} = \bigcap_{\mathfrak{p} \in V(\mathfrak{a})} \mathfrak{p}.$$

Proof. Since $\sqrt{\mathfrak{p}} = \mathfrak{p}$ for any prime ideal $\mathfrak{p}$, it follows that $\sqrt{\mathfrak{a}} \subseteq \bigcap_{\mathfrak{p} \in V(\mathfrak{a})} \mathfrak{p}$. Let $f \notin \sqrt{\mathfrak{a}}$. Then $a \equiv f \bmod \mathfrak{a}$ is not a nilpotent element of $A/\mathfrak{a}$; hence, $\operatorname{Spec}(A/\mathfrak{a})_a$ is not empty by Lemma 1.1. By $V(\mathfrak{a}) \cap D(f) \approx D(a) \approx \operatorname{Spec}(A/\mathfrak{a})_a$, we have $\mathfrak{p}_1 \in V(\mathfrak{a}) \cap D(f)$; hence $\mathfrak{p}_1 \supseteq \mathfrak{a}$ and $f \notin \mathfrak{p}_1$. Thus $f \notin \bigcap_{\mathfrak{p} \in V(\mathfrak{a})} \mathfrak{p}$. □

Corollary. Let $\mathfrak{a}$ and $\mathfrak{b}$ be ideals of A.

(i) $V(\mathfrak{a}) \subseteq V(\mathfrak{b})$ *if and only if* $\sqrt{\mathfrak{a}} \supseteq \sqrt{\mathfrak{b}}$.
(ii) $V(\mathfrak{a}) = V(\mathfrak{b})$ *if and only if* $\sqrt{\mathfrak{a}} = \sqrt{\mathfrak{b}}$.
(iii) $V(\mathfrak{a}) = \operatorname{Spec} A$ *if and only if* $\mathfrak{a} \subseteq \sqrt{(0_A)}$.
(iv) $D(f) = \varnothing$ *if and only if f is nilpotent.*

PROOF. All the assertions follow immediately from Proposition 1.3. □

Proposition 1.4. *Let $\varphi: A \to B$ be a homomorphism of rings. Then for any ideal $\mathfrak{b}$ of B,*

(i) *The closure of* ${}^a\varphi(V(\mathfrak{b}))$ *is* $V(\varphi^{-1}(\mathfrak{b}))$.
(ii) ${}^a\varphi(\operatorname{Spec} B)$ *is dense in* $V(\operatorname{Ker} \varphi)$.
(iii) ${}^a\varphi$ *is dominating (i.e.,* ${}^a\varphi(\operatorname{Spec} B)$ *is dense in* $\operatorname{Spec} A$*) if and only if* $\operatorname{Ker} \varphi \subseteq \sqrt{(0_A)}$.

PROOF. (i) If ${}^a\varphi(V(\mathfrak{b})) \subseteq V(f)$ for some $f \in A$, then any $\mathfrak{q} \in V(\mathfrak{b})$ satisfies ${}^a\varphi(\mathfrak{q}) = \varphi^{-1}(\mathfrak{q}) \ni f$; hence $\mathfrak{q} \ni \varphi(f)$. But since $\sqrt{\mathfrak{b}} = \bigcap_{\mathfrak{q} \in V(\mathfrak{b})} \mathfrak{q}$ by Proposition 1.3, one has $\sqrt{\mathfrak{b}} \ni \varphi(f)$ and so $\varphi^{-1}(\sqrt{\mathfrak{b}}) \ni f$; hence

$$V(f) \supseteq V(\varphi^{-1}(\sqrt{\mathfrak{b}})) = V(\varphi^{-1}(\mathfrak{b})).$$

This implies that the closure of ${}^a\varphi(V(\mathfrak{b}))$ includes $V(\varphi^{-1}(\mathfrak{b}))$. By Proposition 1.1.(ii), we obtain the assertion.

(ii) This follows immediately from (i).

(iii) By assertion (ii), ${}^a\varphi$ is dominating if and only if $V(\operatorname{Ker} \varphi) = \operatorname{Spec} A$. By Corollary (iii) to Proposition 1.3, $V(\operatorname{Ker} \varphi) = V(0_A)$ if and only if $\operatorname{Ker} \varphi \subseteq \sqrt{(0_A)}$. □

Corollary. *For any $f \in A$, $D(f)$ is dense in* $\operatorname{Spec} A$ *if and only if* $I(f) \subseteq \sqrt{(0_A)}$.

PROOF. Since ${}^a\psi_f$: $\operatorname{Spec} A_f \to \operatorname{Spec} A$ is a homeomorphism onto $D(f)$, we can apply Proposition 1.4. □

§1.4 Rings and Modules of Fractions

a. Let M be an A-module and S be a multiplicative subset of A (i.e., $1 \in S$, and $st \in S$ whenever $s, t \in S$). We want to construct the most general A-module N such that for any $s \in S$ and $b \in N$, $sx = b$ is solvable with $x \in N$.

On the Cartesian product $S \times M$, we define the following relation:

$$(s, m) \sim (s', m') \Leftrightarrow t(s'm - sm') = 0 \quad \text{for some } t \in S.$$