Matrices and Graphs

Theory and Applications to Economics

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Theory and Applications to Economics

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Theory and Applications to Economics

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FOREWORD

the editors

The idea to publish this book was born during the conference ≪ Matrices and Graphs: Computational Problems and Economic Applications≫ held in the far June 1993 at Brescia University. The conference was such a success that the organizers, actually the present editors themselves, after a short talk with the lecturers, decided on the spot to apply to the Italian Consiglio Nazionale delle Ricerche (CNR) for a contribution to publish the conference proceedings. The second editor did and the contribution came after a while.

In the meantime the editors organized another conference on ≪Matrices and Graphs: Theory and Economic Applications≫, held as the previous in Brescia during June 1995, partly with different invited lecturers. The conference was a success again and therefore the first editor applied to the Italian National Research Council and got a second contribution, that came only recently.

While the lecturers of the first conference, who were not at the second one, were a bit upset, having submitted their paper without seeing any proceedings published at that time, the lecturers common to the first and the second conference suggested to join the contributions and publish a unique book for both conferences. This is what we did.

During all these years, both authors were very busy lecturing, researching, publishing, raising more funds to make their research possible. Most papers arrived late and were carefully read by the editors, then the search of suitable referees was not easy, so that the reviewing process took also a while, some papers being sent back to the authors for corrections and then submitted again to referees. A complete re-editing was necessary in order to get the uniform editor's style..., well, these are the reasons of such a delay, but eventually here we are.

The book reflects our scientific research background: for academic and scientific reasons both of us were drawn to different research subjects, both shifting from pure to applied mathematics and statistics, with particular attention to data analysis in many different fields the first editor, and to operational research and mathematical finance the second. So, in each of the steps of this long way, we collected a bit of knowledge.

The fact that in most of investigations we dealt with matrices and graphs suggested us to investigate in how many different situations they may be used.

This was the reason that led to the conferences; as a result, this book looks like a patchwork, as it is composed of different aspects: we submit it to the readers, hoping that it will be appreciated, as we did.

In fact, the numerous contributions come from pure and applied mathematics, operations research, statistics, econometrics. Roughly speaking, we can divide the contributions by areas: *Graphs and Matrices*, from theoretical results to numerical analysis, *Graphs and Econometrics*, *Graphs and Theoretical and Applied Statistics.*

Graphs and Matrices contributions begin with **John Maybee**: in his paper *New Insights on the Sign Stability Theorem*, he finds a new characterization of a sign stable matrix, based on some properties of the eigenvectors associated to a sign semi-stable matrix. **Szolt Tuza** in *Lower Bounds for a Class of Depth-Two Switching Circuits* obtains a lower bound for a certain class of (0,1) matrices. It is interesting to note that the problem can be formulated in terms of a semicomplete digraph D, if one wants to determine the smallest sum of the number of vertices in complete bipartite digraphs, whose union is the digraph D itself. **Tiziana Calamoneri** and *Rossella Petreschi*'s *Cubic Graphs as Model of Real Systems* is a survey on cubic graphs, i.e. regular graphs of degree three, and at most cubic graphs, i.e. graphs with maximum degree three and show a few applications in probability, military problems, and financial networks. **Silvana Stefani** and **Anna Torriero** in *Spectral Properties of Matrices and Graphs* describe from one hand how to deduce properties of graphs through the spectral structure of the associated matrices and on the other how to get information on the spectral structure of a matrix through associated graphs. New results are obtained towards the characterization of real spectrum matrices, based on the properties of the associated digraphs. **Guido Ceccarossi** in *Irreducible Matrices and Primitivity Index* obtains a new upper bound for the primitivity index of a matrix through graph theory and extends this concept to the class of periodic matrices. **Sergio Camiz** and **Vanda Tulli** in *Computing Eigenvalues and Eigenvectors of a Symmetric Matrix: a Comparison of Algorithms* compare Divide et Impera, a new numerical method for computing eigenvalues and eigenvectors of a symmetric matrix, to more classical procedures. Divide et Impera is used to integrate those procedures based on similarity transformations at the step in which the eigensystem of a tridiagonal matrix has to be computed.

Among contributions on *Graphs and Econometrics* we find **Sergio Camiz** paper *I/O Analysis: Old and New Analysis Techniques.* In this paper, Camiz compares various techniques used in I/O analysis to reveal the complex structure of linkages among economic sectors: triangularization, linkages comparison, exploratory correspondence analysis, etc. Graph analysis, with such concepts as centrality, connectivity, vulnerability, turns out to be a useful tool for identifying the main economic flows, since it is able to reveal the most important information contained in the I/O table. **Manfred Gilli** in *Graphs and Macroeconometric Modelling* deals with the search of a local unique solution of a system of equations and with necessary and sufficient conditions for this solution to hold. He shows how through a graph theoretic approach the problem can be efficiently investigated, in particular when the Jacobian matrix is large and sparse, a typical case of most econometric models. **Manfred Gilli** and **Giorgio Pauletto** in *Qualitative Sensitivity Analysis in Multiequation Models* perform a sensitivity analysis of a given model when a linear approximation is used, the sign is given and there are restrictions on the parameters. They show that a qualitative approach, based on graph theory, can be fruitful and lead to conclusions which are more general than the quantitative ones, as they are not limited to a neighborhood of the particular simulation path used. **Mario Faliva** in *Hadamard Matrix Product, Graph and System Theories: Motivations and Role in Econometrics* shows how the analysis of a model's causal structure can be handled by using Hadamard product algebra, together with graph theory and system theoretical arguments. As a result, efficient mathematical tools are developed, to reveal the causal and interdependent mechanisms associated with large econometric models. At last, *International Comparisons and Construction of Optimal Graphs*, by **Bianca Maria Zavanella**, contains an application of graph theory to the analysis of the European Union countries based on prices, quantities and volumes. Graph theory turns out to be a most powerful tool to show which nations are more similar.

Graphs and Statistics papers are represented by three contributions. **Giovanna Iona Lasinio** and **Paola Vicard** in *Graphical Gaussian Models and Regression* review the use of graphs in statistical modelling. The relative merits of regression and graphical modelling approach are described and compared both form the theoretical point of view and with application to real data. **Francesco Lagona** in *Linear Structural Dependence of Degree One among Data: a Statistical Model models* the presence of some latent observations using a linear structural dependence among data, thus deriving a particular Markovian Gaussian field. **Bellacicco** and **Tulli** in *Cluster identification in a signed graph by eigenvalue analysis* establish a connection between clustering analysis and graphs, by including clustering into the wide class of a graph transformation in terms of cuts and insertion of arcs to obtain a given topology.

After this review, it should be clear how important is the role of matrices and graphs and their mutual relations, in theoretical and applied disciplines. We hope that this book will give a contribution to this understanding.

We thank all the authors for their patience in revising their work. A special thanks goes to Anna Torriero and Guido Ceccarossi for their constant help, but especially we would like to thank Vanda Tulli, who did the complete editing trying to (and succeeding in) making order among the many versions of the papers we got during the revision process. Last, but not least, thanks to Mrs. Chionh of World Scientific Publishers in Singapore, whom we do not know personally, but whose efficieny we had the opportunity to know through E-mail.

October, 1996.

Sergio Camiz and Silvana Stefani

The manuscripts by Sergio Camiz, Guido Ceccarossi, Manfred Gilli, Giovanna Iona Lasinio and Paola Vicard, Francesco Lagona, and Bianca Maria Zavanella, referring to the first Conference, have been received at the end of 1993. The manuscripts of Antonio Bellacicco and Vanda Tulli, Tiziana Calamoneri and Rossella Petreschi, Sergio Camiz and Vanda Tulli, Mario Faliva, Manfred Gilli and Giorgio Pauletto, John Maybee, Silvana Stefani and Anna Torriero, and Szolt Tuza, referring to the second Conference, were received at the end of 1995.

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Contents

NEW INSIGHTS ON THE SIGN STABILITY PROBLEM

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We obtain a new characterization of when a matrix is sign stable. Our results makes use of properties of eigenvectors of sign semi-stable matrices. No classical stability theorems are required in proving our results.

1 Introduction

We deal with $n \times n$ real matrices. Such a Matrix A is called *semistable (stable)* if every λ in the spectrum of $A, \sigma(A)$ lies in the closed (open) left-half of the complex plane. The real matrix $\text{sgn}(A) = [\text{sgn } a_{ij}]$ is called the *sign pattern* of A and two real matrices A and B are said to have *the same sign pattern* if either $a_{ij}b_{ij} > 0$ or both a_{ij} and b_{ij} are zero for all i and j. When A is a real matrix we let $Q(A)$.be the set of all matrices having the same sign pattern as A. We also will write A in the form $A = A_d + \tilde{A}$ where $A_d = \text{diag}[a_{11}, a_{22}, \ldots, a_{nn}]$ and $\tilde{A} = A - A_d$.

Let u be a complex vector $u = (u_1, u_2, \ldots, u_n)$. We say that u is *q-orthogonal to A_d* if $a_{ii} \neq 0$ implies $u_i = 0$. Notice that if u is *q-orthogonal* to A_d, then u is *q-orthogonal* to B_d for every matrix $B \in Q(A)$.

Let A be a real matrix satisfying $a_{ij} \neq 0$ if and only if $a_{ji} \neq 0$ for all $i \neq j$. Then A is called *combinatorially symmetric* and we may associate with A the *graph* $G(A)$ having n vertices and an edge joining vertices i and j if and only if $i \neq j$ and $a_{ij} \neq 0$. The graph $G(A)$ is a *tree* if it is connected and acyclic. We also use, for any matrix, the directed graph $D(A)$ defined in the usual way.

The real matrix A is called sign *semi-stable (sign-stable)* if every matrix in $Q(A)$ is semi-stable (stable). We will deal only with the case where A is irreducible in order to keep the arguments simple (Gantmacher, 1964). All of our results can be readily extended to the reducible case.

We will prove the following results about sign semi-stable matrices.

Theorem 1 *The following are equivalent statements:*

1. *The matrix A is sign semi-stable.*

2. *Matrix A satisfies*

   (i) $a_{jj} \leq 0$, *for all* j,

(ii) $a_{ij}a_{ji} \leq 0$ *for all* i *and* j, *and*

(iii) *every product of the form* $a_{i(1)i(2)}a_{i(2)i(3)} \cdots a_{i(k)i(1)} = 0$ *for* $k \geq 3$ *where* $\{i(1), i(2), \ldots, i(k)\}$ *is a set of distinct integers in* $N = \{1, 2, \ldots, n\}$.

3. *There exists a positive diagonal matrix* $D = diag[d_1, d_2, \ldots, d_n]$, $d_i > 0$, $i = 1, 2, \ldots, n$ *such that the matrix* $DAD^{-1} = A_d + S$ *where* S *is skew symmetric (Gantmacher, 1964) and satisfies (ii), and (iii).*

Theorem 2 *The following are equivalent statements about a sign semi-stable matrix:*

1′. *The matrix* A *has* $\lambda = 0$ *as an eigenvalue.*

2′. *Every matrix in* $Q(A)$ *has* $\lambda = 0$ *as an eigenvalue.*

3′. *There is an eigenvector* u *satisfying* $Au = 0$ *which is q-orthogonal to* A_d.

Theorem 3 *The following are equivalent statements about a sign semi-stable matrix:*

1″. *The matrix* A *does not have a purely imaginary eigenvalue.*

2″. *No matrix in* $Q(A)$ *has a nonzero purely imaginary eigenvalue.*

3″. *There is no eigenvector* u *satisfying* $Au = i\mu u$, *which is q-orthogonal to* A_d.

The equivalence of conditions (1) and (2) of Theorem 1 is a well known result due to Maybee, Quirk, and Ruppert (see Jefferies *et al*, 1977 for one proof of this result). All the known proofs of this equivalence make use of one of the classical stability theorems. By proving that (1) ⇒ (2) ⇒ (3) ⇒ (1) we can avoid the use of any stability theorem, a fact of some independent interest.

A consequence of Theorem 1 is that the family of sign semi-stable matrices can be identified with the family of matrices of the form $A = A_d + S$, where S is skew-symmetric and A satisfies (i),(ii), and (iii). This fact is used in an essential way to prove Theorem 3.

Our proofs of Theorems 2 and 3 lead directly to simple algorithms for testing a given matrix satisfying the conclusions (i),(ii), and (iii) to determine whether or not it is sign-stable.

Finally, given Theorems 1,2, and 3 we can state the following sign stability result.

Theorem 4 *The real matrix A is sign stable if and only if the following four conditions are satisfied.*

(i) $a_{jj} \leq 0$ *for all* j;

(ii) $a_{ij}a_{ji} \leq 0$ *for all* i *and* j;

(iii) every product of the form $a_{i(1)i(2)}, a_{i(2)i(3)} \cdots a_{i(k)i(1)} = 0$ *for* $k \geq 3$ *where* $\{i(1), i(2), \ldots, i(k)\}$ *is a set of distinct integers in* $N = \{1, 2, \ldots, n\}$

(iv) the matrix A *does not have an eigenvector q-orthogonal to* A_d

2 The proof of Theorem 1

Suppose first that the matrix A is sign semi-stable. The fact that (i),(ii), (iii), are then true follows by a familiar continuity argument which we omit. Hence (1) $\Rightarrow$ (2). Given that (2) is true and A is irreducible it follows that, if $a_{ij} \neq 0$ then $a_{ji} \neq 0$ also. For suppose $a_{ij} \neq 0$ and $a_{ji} = 0$. Since there is a path from j to i in A, (iii) is violated. Thus A is combinatorially symmetric. But then $G(A)$, the graph of A exists, is connected and has no cycles. Hence $G(A)$ is a tree. Then by a theorem of Parter and Youngs (1962), there exists a positive diagonal matrix D such that $DAD^{-1} = A_d + S$ where $A_d = \text{diag}[a_{11}, a_{22}, \ldots, a_{nn}], S = [S_{ij}]$, with $S_{ii} = 0$ for $i = 1, 2, \ldots, n$, and $S_{ij} = -S_{ji}$ for all $i \neq j$. Thus (2) $\Rightarrow$ (3). Now set $\hat{A} = DAD^{-1}$ and suppose $\hat{A}u = \lambda u$. Taking scalar products on the right and left with u yields $u \cdot \hat{A}u = u \cdot A_d u + u \cdot Su = u \cdot \lambda u = \bar{\lambda}|u|^2$ and $\hat{A}u \cdot u = A_d u \cdot u + Su \cdot u = \lambda|u|^2$. We have $u \cdot A_d u = A_d u \cdot u$ and $u \cdot Su = -Su \cdot u$. Hence $2A_d u \cdot u = (\lambda + \bar{\lambda})|u|^2$ so we obtain

$$Re(\lambda) = \frac{A_d u \cdot u}{|u|^2} \tag{1}$$

But $A_d u \cdot u = \sum_{i \in I_0} a_{ii}|u|^2$ where $I_0 = \{j \mid a_{jj} \neq 0\}$. Hence condition (i) implies that for any λ in $\sigma(A)$, $Re(\lambda) \leq 0$. Thus (3) implies (1) and Theorem 1 is proved.

Now a sign semi-stable matrix is sign stable if and only if it has no eigenvalues on the imaginary axis in the complex plane. On the other hand, if u is an eigenvector of A belonging to an eigenvalue on the imaginary axis, then we must have $u_i = 0$ if $i \in I_0$ by (1), i.e. u is q-orthogonal to A_d.

Note also that it follows from the proof of Theorem 1 that, if $a_{ii} = 0, i = 1 \ldots n$, A is skew-symmetric and all the eigenvalues of A are purely imaginary, hence A is not sign stable. If $a_{ii} < 0,\ i = 1 \ldots n$, then every nonzero vector

u satisfies $Re(\lambda) < 0$ so A is sign stable. Hence the interesting case for sign stability is $1 \leq |I_0| < n$, which we assume to hold henceforth.

3 The proof of Theorem 2

Let A be a sign semi-stable matrix. By conditions (i) and (ii) every term in the expansion of $\det A$ has the same sign. Therefore if $\det A = 0$, it must be combinatorially equal to zero and hence every matrix in $Q(A)$ also has determinant equal to zero. It follows that (1′) implies (2′). Our task is to discover when there exists a non-zero vector u such that $Au = 0$. Now u must vanish on the (nonempty) set I_0 so we partition the components of a candidate vector u initially into the sets $Z(I_0), N(I_0)$ where $Z(I_0) = \{i \mid i \in I_0\}, u_i = 0$ if $i \in Z(I_0)$, and $u_i \neq 0$ if $i \in N(I_0)$. Now given a set $I_p \supseteq I_0$ and a partition of the components of u such that $u_i = 0$ if $i \in Z(I_p)$ and $u_i \neq 0$ if $i \in N(I_p)$. We look at the equations

$$\sum_{j \in N(I_p)} s_{ij} u_j = 0 \tag{2}$$

If such an equation has exactly one nonzero term, it has the form $s_{ik}u_k = 0$ for some fixed value of k. Since $s_{ik} \neq 0$ and $k \in N(I_p)$, this is a contradiction. Hence we must place $k \in I_{p+1}$. We do this for each such occurrence. Thus $I_{p+1} \supseteq I_p$ and $Z(I_p) \subseteq Z(I_{p+1}), N(I_p) \supseteq N(I_{p+1})$. If the system (2) contains no equation having only a single non-zero term, then $I_{p+1} = I_p$ and $Z(I_{p+1}) = Z(I_p), N(I_{p+1}) = N(I_p)$. We will examine this case below. Suppose that $I_{p+1} = N$. Then $Z(I_{p+1}) = N$ and u=0, i.e. no matrix in $Q(A)$ has zero as an eigenvalue. It remains to consider the case where we have some $I_p = I_{p+1}$ with $|I_p| < n$ so $u_i = 0$ for $i \in Z(I_p)$ and $u_i \neq 0$ for $i \in N(I_p)$. Clearly every equation in system (2) at this point contains either no non-zero terms or at least two non-zero terms. We have $N(I_p) \geq 2$ and the induced graph $\langle N(I_p) \rangle$ is a forest. We claim that this forest consists of isolated single trees, i.e. $S(N(I_p)) = 0$. For suppose $\langle N(I_p) \rangle$ has a nontrivial tree T_0. This tree has a vertex of degree one and there would then exist an equation in the subsystem $S(N(I_p))u = 0$ having exactly one nonzero term, a contradiction. Next let $|N(I_p)| = q$ and suppose there exists r rows in the subsystem $\sum_{u \in N(I_p)} s_{ij} u_i = 0$, $i \in Z(I_p)$, having two or more non-zero entries. We have r ≥ 1 so the set of such rows is nonempty. Let this set be $Z_0(I_p)$ and consider the submatrix $S(N(I_p)) \cup Z_0(I_p)$. The graph of this submatrix is a forest on the $q + r$ vertices. If $|Z_0(I_p)| \geq q$ then the numbers of edges in this forest is at least $2r \geq r + q$, a contradiction. Similarly, there cannot be two directed paths from vertex k to