FOUNDATIONS OF ABSTRACT MATHEMATICS

DAVID C. KURTZ

INTERNATIONAL SERIES IN PURE AND APPLIED MATHEMATICS

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Rollins College

McGraw-Hill, Inc.
New York St. Louis San Francisco Auckland Bogotá
Caracas Lisbon London Madrid Mexico Milan
Montreal New Delhi Paris San Juan Singapore
Sydney Tokyo Toronto

This book was set in Times Roman by Publication Services.
The editors were Richard Wallis and Margery Luhrs;
the production supervisor was Denise L. Puryear.
The cover was designed by Karen K. Quigley.
R. R. Donnelley & Sons Company was printer and binder.

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2 3 4 5 6 7 8 9 0 DOH DOH 9 0 9 8 7 6 5 4 3 2

ISBN 0-07-035690-4

Library of Congress Cataloging-in-Publication Data

Kurtz, David.
Foundations of abstract mathematics / by David Kurtz.
p. cm. — (International series in pure and applied mathematics)
Includes index.
ISBN 0-07-035690-4
1. Logic, Symbolic and mathematical. 2. Mathematics. I. Title. II. Series.
QA9.K797 1992 91-338
511.3—dc20

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To Judy

PREFACE

One of the most difficult steps a student of mathematics must make is the one into that (blissful) state known as "mathematical maturity." This is a step which is accomplished by making the transition from solving problems in a fairly concrete setting in which there is a well-known method or an algorithm for each problem type (as in most calculus courses, for example) to writing proofs and producing counterexamples involving more abstract objects and concepts, an activity for which there is no well-defined algorithm. Often this transition is something which is expected just to "happen," perhaps during the summer between the sophomore and junior years; however, it is not clear what summertime activities one could recommend to ensure such a result. My recent teaching experience suggests that this transition is not an easy one for most students and generally cannot be successfully made without some concerted effort and guidance. Two things which seem to inhibit a smooth transition are a lack of knowledge of some fundamental mathematical ideas—logic, sets, functions—and a lack of experience in two important mathematical activities—finding examples of objects with specified properties and writing proofs. This book is an attempt to provide an opportunity to gain exposure to these activities while learning some of the necessary fundamental ideas.

I have tried to keep the book as short as possible to achieve these goals; thus some interesting topics are left out and others are treated only in the exercises. I have also tried to take a developmental point of view so that the book starts out in a fairly simple, informal manner and gradually becomes more formal and abstract. This means that while it is possible to cover the first chapter rather rapidly, one should not expect to maintain this speed throughout the book; indeed, I have found that some sections in Chapter 2 can easily take more than a week to cover with any degree of thoroughness.

The transitional process begins with an informal introduction to logic, including a careful consideration of quantifiers and a discussion of basic proof forms. The objective here is to obtain a firm foundation on which to build the proof-writing skills which will be developed later on. The first mathematical objects encountered in any detail are sets. This provides a familiar setting in which students can write simple proofs, test conjectures and produce counterexamples. The next topic, relations, is probably not as familiar as the topic of sets, and here the students get their first taste of trying to understand the definition of a new concept (e.g., equivalence relation, strict partial order) well enough to provide examples and proofs. Functions are presented as special relations and functional composition is emphasized. Chapter 2 concludes with binary operations and equivalence relations induced by functions, a foreshadowing of the fundamental theorem of group homomorphisms. Several forms of mathematical induction are presented in chapter 3, and since the students should have acquired a working knowledge of implications, propositional functions and sets by this time, there is some chance that their understanding of induction will be more than an algorithmic one.

These first three chapters form what I think should be the core of the course; in fact, with some difficulty I have been restrained from calling it "What Every Mathematics Student Should Know." As time permits (and it sometimes actually does), any of the last three chapters may be studied independently of one another in accordance with the interests and needs of the class. Each is reasonably self-contained and chapters 5 (Groups) and 6 (Cardinality) do not require any previous knowledge and have the advantage of presenting material which is new to the students. Chapter 4 (Continuity Carefully Considered) probably should not be attempted by students who have not had a year of calculus. It begins with a development of the real number system, including algebraic order and metric properties. Extensive use is made of sequences in examining the concepts of limits, continuity and uniform continuity. In chapter 5, cosets are discussed in some detail (as examples of partitions) and their connection to homomorphisms is explored. In chapter 6 much use is made of one-to-one correspondences. The properties of finite and infinite sets are distinguished and cardinal numbers are discussed. In each of these "application" chapters no attempt has been made to give a comprehensive view of the subject being considered; rather, a small area has been examined in sufficient depth so that some non-trivial results can be shown (e.g., intermediate value theorem, fundamental theorem of group homomorphisms, the uncountability of $\mathbb{R}$).

One final comment: In days gone by I thought that if I could organize the material to be presented in a cogent fashion, develop the students' interest in it and provide good examples and answers to their questions, I would be a good teacher and they would learn a lot of mathematics. That is, I thought that what I did was the important part of the educational process. Now I have come to believe that what *I* do is not nearly so important as what I can get the *students* to do. This means that it is impossible to overemphasize the importance of having the students do exercises. I have provided a wide selection of exercises, many of which are presented as conjectures to be verified or shown incorrect. A somewhat unusual sort of exercise which appears throughout the last five chapters of the book is the "Believe It or Not" exercise. In these exercises a conjecture is given, along with a proof and a counterexample. Of course, at least one must be incorrect (sometimes all three are), and the student's task is to sort things out and put them right by determining the true state of affairs and giving (if necessary) a correct proof or counterexample and pointing out the errors in the ones given.

I am not sure if it is possible to *teach* someone how to write proofs, any more than it is possible to teach them how to write poetry or compose a symphony. However, I do think that it is possible to help someone *learn* how to write proofs and I hope that this book is useful in accomplishing this important task.

I am grateful to the many students who, over the years, labored through the large number of iterations of this material. In many cases they have inspired the interesting, but incorrect, parts of the "Believe It or Not" exercises. I want to thank the reviewers whose helpful comments have improved the exposition and helped root out unclear passages:

Barbara Bohannon, Hofstra University;
Richard E. Chandler, North Carolina State University;
Harvey Charlton, North Carolina State University;
Charles Clever, South Dakota State University;
Peter Colwell, Iowa State University;
Gary D. Crown, Wichita State University;
Bruce Edwards, University of Florida;
Robert O. Gamble, Winthrop College;
John I. Gimbel, University of Alaska;
Stephen Pennell, University of Lowell.

Of course any inaccuracies and opaqueness remaining are mine. I am also indebted to the staff at McGraw-Hill who have made the production of this book a pleasure.

David C. Kurtz

A FEW WORDS FOR THE READER

Many students have difficulty when they are first asked to prove theorems in mathematics. Part of this difficulty may come from an unfamiliarity with the mathematical objects involved (vectors, bases, linear transformations, groups, homomorphisms, and so forth), but a major part of the difficulty seems to be due to an imprecise knowledge of the fundamentals of mathematics: logic, sets, relations and functions. This book attempts to address this problem by giving a concise account of a minimal amount of this material needed to progress further in mathematics and then using this material as a vehicle for gaining practice in proving theorems.

The key word here is *practice*. As you no doubt have observed, learning how to write out a correct proof yourself is quite a bit different from watching someone else write out a proof and understanding that his or her proof is correct. Mathematics is not a spectator sport! Practice and involvement are essential. If anything is to be gained from this book, the reader must become actively engaged in working his or her way through it. This means marking up the pages with questions about unclear passages (should there be any!), doing the examples and then checking the results, working all the exercises and, above all, approaching the subject matter with a questioning mind intent upon gaining a thorough understanding of it.

A passive approach is doomed to failure. A pencil and paper should be at hand before you start reading. Of course, this means that you won't be able to read 20 pages a night; 3 pages would be a more reasonable goal, especially further along in the book where the level of abstraction is somewhat higher and more is expected of you. But as in anything where a considerable effort is required, the rewards are equally great; the satisfaction of writing a proof which you *know* is correct is hard to match. So pick up your pencil (or pen or whatever it is you use) and proceed at a deliberate pace through the following pages, knowing that mastery of their contents will lead to mathematical pleasures unknown to the uninitiated.

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CHAPTER 1

LOGIC

1.1 INTRODUCTION

A friend of mine recently remarked that when he studied logic he got sleepy. I replied that he looked sleepy at the moment and he said, "Yes, I am sleepy." He added, "Therefore, you can conclude that I have been studying logic." "Most certainly not!" I answered. "That's a good example of an invalid argument. In fact, if you have been studying logic it's obvious that you haven't learned very much."

This short excerpt from a real-life situation is meant to illustrate the fact that we use logic in our everyday lives—although we don't always use it correctly. Logic provides the means by which we reach conclusions and establish arguments. Logic also provides the rules by which we reason in mathematics, and to be successful in mathematics we will need to understand precisely the rules of logic. Of course, we can also apply these rules to areas of life other than mathematics and amaze (or dismay) our friends with our logical, well-trained minds.

In this chapter we will describe the various connectives used in logic, develop some symbolic notation, discover some useful rules of inference, discuss quantification and display some typical forms of proof. Although our discussion of connectives and truth tables in the beginning is rather mechanistic and does not require much thought, by the end of the chapter we will be analyzing proofs and writing some of our own, a very non-mechanistic and thoughtful process.

1.2 AND, OR, NOT, AND TRUTH TABLES

The basic building blocks of logic are *propositions*. By a proposition we will mean a declarative sentence which is either true or false but not both. For example, "2 is greater than 3" and "All equilateral triangles are equiangular" are propositions while "$x < 3$" and "This sentence is false" are not (the first of these is a declarative sentence but we cannot assign a truth value until we know what "x" represents; try assigning a truth value to the second). We will denote propositions by lowercase letters, p, q, r, s, etc. In any given discussion different letters may or may not represent different propositions but a letter appearing more than once in a given discussion will always represent the same proposition. A true proposition will be given a truth value of T (for true) and a false proposition a truth value of F (for false). Thus "$2 + 3 < 7$" has a truth value of T while "$2 + 3 = 7$" has a truth value of F.

We are interested in combining simple propositions (sometimes called *subpropositions*) to make more complicated (or compound) propositions. We combine propositions with *connectives*, among which are "and," "or" and "implies." If p, q are two propositions then "p and q" is also a proposition, called the *conjunction* of p and q, and denoted by

$$p \wedge q.$$

The truth value of $p \wedge q$ depends on the truth values of the propositions p and q: $p \wedge q$ is true when p and q are both true, otherwise it is false. Notice that this is the usual meaning we assign to "and." The word "but" has the same logical meaning as "and" even though in ordinary English it carries a slightly different connotation. A convenient way to display this fact is by a *truth table*. As each of the two propositions p, q has two possible truth values, together they have $2 \times 2 = 4$ possible truth values so the table below lists all possibilities:

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

Thus, for example, when p is T and q is F (line 2 of the truth table), $p \wedge q$ is F. In fact, this truth table can be taken as the definition of the connective $\wedge$.

It should be noted here that the truth table above does not have anything to do with p and q; they are just placeholders—cast in the same role as x in the familiar functional notation $f(x) = 2x - 3$. What the truth table does tell us, for example, is that when the first proposition is F and the second is T (third row of the table) the conjunction of the two propositions is F. You can check your understanding of this point by working exercise 5 at the end of this section.

Another common connective is "or," sometimes called *disjunction*. The disjunction of p and q, denoted by

$$p \vee q$$

is true when *at least one* of p, q is true. This is called the "inclusive or"; it corresponds to the "and/or" sometimes found in legal documents. Note that in ordinary conversation we often use "or" in the exclusive sense; true only when *exactly one* of the subpropositions is true. For example, the truth of "When you telephoned I must have been in the shower or walking the dog" isn't usually meant to include both possibilities. In mathematics we always use "or" in the inclusive sense as defined above and given in the truth table below:

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

Given any proposition p we can form a new proposition with the opposite truth value, called the *negation of p*, which is denoted by

$$\neg p.$$

This is sometimes read as "not p."

The truth table for negation is

p	$\neg p$
T	F
F	T
.

We can form the negation of a proposition without understanding the meaning of the proposition by prefacing it with "it is false that" or "it is not the case that" but the resulting propositions are usually awkward and do not convey the real nature of the negation. A closer consideration of the meaning of the proposition in question will often indicate a better way of expressing the negation; later we will consider methods for negating compound propositions.

Consider the examples below:

a) $3 + 5 > 7$.

b) It is not the case that $3 + 5 > 7$.

c) $3 + 5 \leq 7$.

d) $x^2 - 3x + 2 = 0$ is not a quadratic equation.

e) It is not true that $x^2 - 3x + 2 = 0$ is not a quadratic equation.

f) $x^2 - 3x + 2 = 0$ is a quadratic equation.

Note that b) and c) are negations of a); e) and f) are negations of d), but c) and f) are to be preferred over b) and e), respectively.

We will use the same convention for $\neg$ as we use for $-$ in algebra; that is, it applies only to the next symbol, which in our case represents a proposition. Thus $\neg p \vee q$ will mean $(\neg p) \vee q$ rather than $\neg(p \vee q)$, just as $-3 + 4$ represents 1 and not -7. With this convention we can be unambiguous when we negate compound propositions using symbols, but life is not so easy when we consider how to negate compound propositions in English. For example, how do we distinguish between $\neg p \vee q$ and $\neg(p \vee q)$ in English? Suppose p represents "$2 + 2 = 4$," and q represents "$3 + 2 < 4$." Should "It is not the case that $2 + 2 = 4$ or $3 + 2 < 4$" mean $\neg(p \vee q)$ or $\neg p \vee q$? If we use the same convention we used for our symbols it should mean $\neg p \vee q$. But, if we take this meaning, then how would we say $\neg(p \vee q)$? The problem seems to be a lack of the equivalent of the parentheses we used for grouping. Let us adopt the convention that "it is not the case that" (or a similar negating phrase) applies to everything that follows, up to some sort of grouping punctuation. Thus, "It is not the case that $2 + 2 = 4$ or $3 + 2 < 4$" would mean $\neg(p \vee q)$, while "It is not the case that $2 + 2 = 4$, or $3 + 2 < 4$" would mean $\neg p \vee q$. Of course, when speaking, one must be very careful about using pauses to indicate the proper meaning.

Truth tables can be used to express the possible truth values of compound propositions by constructing the various columns in a methodical manner. For example, suppose that we wish to construct the truth table for $\neg(p \vee \neg q)$. We begin by making a basic four-row (there are four possibilities) truth table with column headings:

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