Algebraic and Arithmetic Structures

A Concrete Approach For Elementary School Teachers

Max S. Bell Karen C. Fuson Richard A. Lesh

algebraic and arithmetic structures

A CONCRETE APPROACH FOR ELEMENTARY SCHOOL TEACHERS

developed cooperatively by

MAX S. BELL
The University of Chicago

KAREN C. FUSON
Northwestern University

RICHARD A. LESH
Northwestern University

THE FREE PRESS
A Division of Macmillan Publishing Co., Inc.
NEW YORK

Collier Macmillan Publishers
LONDON

The Free Press
A Division of Macmillan Publishing Co , Inc
866 Third Avenue, New York, N Y 10022

Collier Macmillan Canada, Ltd

Library of Congress Catalog Card Number 75-2807

Printed in the United States of America

printing number
1 2 3 4 5 6 7 8 9 10

Library of Congress Cataloging in Publication Data

Bell, Max S
Algebraic and arithmetic structures.

Bibliography: p.
Includes index.
1. Mathematics--Study and teaching (Elementary)
I. Fuson, Karen C., joint author. II. Lesh, Richard A., joint author. III. Title.
QA135.5.B43 372.7 75-2807
ISBN 0-02-902270-3

preface

A few years ago at the University of Chicago and at Northwestern University we found that our lectures on mathematics were having only limited success in teaching prospective elementary school teachers. Our students often came to us with deep-seated fears and virtually no self-confidence with respect to mathematics. Yet they were well aware that a year or so hence they would be teaching mathematics. They wanted practical, sure-fire prescriptions for accomplishing what they were sure would be an unpleasant task, and their fears and anxieties often left little tolerance for our excursions into theoretical mathematics, however "simple" and "nice." On our part, we felt it would be unfair to them and their future pupils if they didn't acquire an understanding of mathematics well beyond that contained in the school books they would use.

As we struggled with such dilemmas, we began first to use such things as counters and rods when they could make some obvious contribution to the theory we wanted to teach. Encouraged by the success of this, we began actively to search for ways of illustrating whatever mathematics was at hand by activity-oriented exercises. We soon came to believe that it is possible to find or invent helpful "concretizations" or "embodiments" for practically every concept of elementary mathematics. That is, an activities oriented approach imposed virtually no restrictions on the mathematics content we believe teachers should know. Furthermore, with these activities, adults were willing to tackle once again mathematical topics that had often been mysterious or threatening to them. They saw the activities as suggesting ways to teach similar concepts to youngsters, and this provided additional motivation. Eventually we began to wish to share these rather nice results with our colleagues, and this book is the result.

No particular course or method is likely to suit every student, yet we have found the range of students to whom these materials appeal to be rather surprising. This range has included the usual college age students and mature students returning to college after many years; students with considerable and with little mathematical training; students at relatively "high-power" private universities, and those attending urban commuter universities, and pre-service teachers and in-service teachers. Depending on mathematical expertness and experience, the material in this book either allows a fresh start and gentle guidance in coming to terms with some mathematics, or a reexamination of fundamental concepts which were never difficult for a given teacher but which must now be taught to youngsters who have considerable difficulty.

The material contained herein is an adult level treatment of the arithmetic and algebra of whole numbers, integers, rational numbers, and (briefly) real numbers plus some excursions into theoretical structures involving material on logic, number theory, relations, functions, and modern algebra. The material has no mathematical prerequisites since every topic is started from scratch in quite accessible ways.

In working with those adults who are reconstructing many of their concepts of mathematics (and perhaps their mathematical confidence as well), several things are often mentioned by them as impressions taught by their own school mathematics experience. First, their school experience has often led them to the false belief that mathematics is a completed rather than a growing and developing field of inquiry. Second, they have been given little reason to believe that mathematics is good for anything, except possibly the everyday uses of very simple calculation. Third, they have often come to think of mathematics as tricky symbol manipulation using some sort of magic, in which "correct" results are more a matter of good luck than of good management. They frequently express some anxiety about being urged to work things out first in action oriented, mostly nonsymbolic ways that seem to come from a feeling that it is almost *immoral* not to do things the hard way; or that using concrete materials is an indication that one is too stupid to do it the "right" way.

Such misconceptions about mathematics itself, the usefulness of mathematics, and the variety of respectable ways in which mathematics can be learned (or indeed created) are most unfortunate. This book is motivated principally by the hope of breaking into the circle in which teachers with such misconceptions pass them on to children.

We had help from many sources in preparing this book, our families not least of all. Special thanks are due Barry Hammond for early help and encouragement, Hazel Wagner for assistance in developing some of the materials and Warren Crown, Sandy Kerr, Michael Mahaffey, and Lauren Woodby for helpful criticism of various drafts. Thanks also go to our typists and to Robert Garfield, Michael Harkavy, and Ellen Simon of The Free Press.

Chicago and Evanston, Illinois
December 1975

KF
RL
MB

introduction

0.1. ASSUMPTIONS INFLUENCING THE CONTENT OF THIS BOOK

1. We are convinced that in spite of years of exposure to school mathematics, many adults (perhaps the majority) have negative feelings toward the subject and are often unable to *use* mathematics comfortably. This means to us that mathematics education has failed for far too many people. It seems to us that the most likely way to change this state of affairs is to provide a much better mathematics experience in the early school years. In order to do this, elementary school teachers must be much better prepared—and prepared in a new way—for teaching mathematics.
2. To us, better teacher preparation means at least two things. First, teachers must become much more expert in mathematics itself; hence, our selection of content in this book comes from good solid mathematical ideas and structures. Second, teachers need to consider deeply and carefully how they can help children build for themselves durable mathematical intuitions, concepts, and skills.
3. With possible (but rare) exceptions, mathematics cannot simply be poured into a person's head in its purest and best symbolic form; most people must build it for themselves, starting from considerable concrete experience. If that is so, the way in which mathematics is learned assumes importance along with what is learned. Hence, although this is a course in mathematics, it illustrates some important "methods" by which good mathematics can be taught.
4. We believe that nearly all of school mathematics is (or should be) "useful" mathematics, but that this usefulness is often obscured or ignored in school instruction. Hence, as you go through this book we encourage you to keep asking, in effect, "Why bother? Who needs it?" [1]
5. Contrary to the impression one might get from much school instruction, mathematics is much more than computation. What we ultimately want is an understanding of mathematical operations and structures along with considerable flexibility and ingenuity in using mathematics.

[1] You should come to expect answers of two sorts to such a question The first involves a need for new mathematical tools to cope with the actual real world For example, fractions are needed as soon as one must cope with measures. The second involves the need to tidy up or extend basic mathematical structures. For example, negative numbers need inventing in order that all subtraction problems will have answers (e g , $3 - 23 = -20$.) (Negative numbers are also useful in the real world, of course)

This book, then, aims at an understanding of computation, but even more it is designed to help each person build a sure intuitive feel for numbers and operations ("number sense"); build an equally secure "measure sense"; build an understanding of the several uses of variables; and combine these into the problem-solving skills that allow one to look at a new situation, size it up, and *do* something with it.

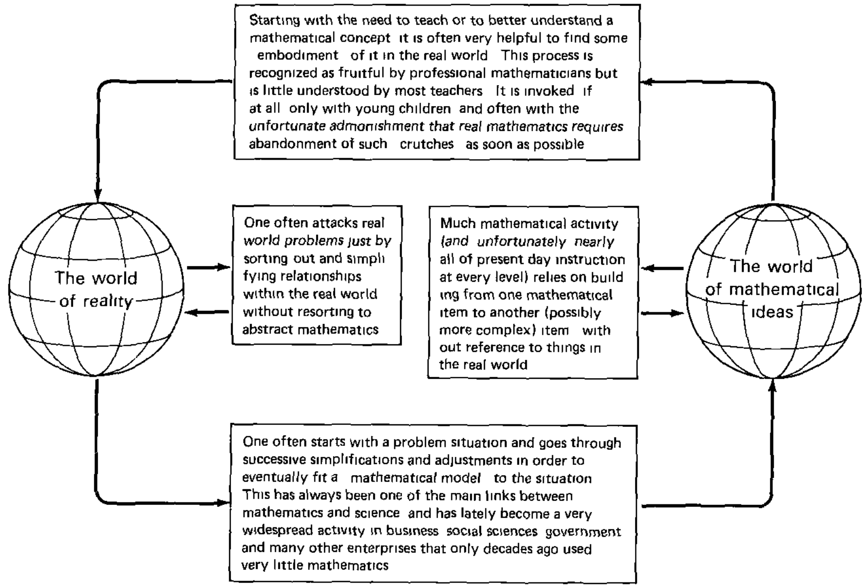
6. *Some part* of a person's mathematical knowledge does involve computation. In this respect, we believe that an understanding of what the various operations do, and how they behave, is a distinct issue from consideration of the sometimes complicated algorithms of arithmetic (e.g., long division). For example, one might know that for a given problem dividing 375 by 23 is the appropriate thing to do without being able to carry out the algorithmic process to get an answer. And, of course, one can be a whiz at the long division algorithm but understand little about division itself.[2] Hence, the first third of this book is about number systems and what the operations mean in each; the middle third of the book deals with algorithmic processes, which we believe are interesting and important in their own right; and the final third of the book deals in a more general way with underlying algebraic ideas that were considered in the other two-thirds of the book. The net result is an adult-level, relatively advanced consideration of arithmetic and algebra that uses activities and applications in most of the learning sequences.

0.2. MATHEMATICS AND REAL WORLD LINKS

Such beliefs as those outlined above have led to consistent emphasis throughout the book on several major themes: applications of mathematics; "embodiment" or "concretization" of theoretical mathematics by actions on concrete materials; mathematical structures. All these touch in one way or another on links and interrelationships between the "world of mathematics" and the "world of reality." The range of possibilities is indicated by the diagram on the facing page.

Neither "mathematical models" nor "embodiments" of mathematics have been emphasized in mathematical instruction in the past, and even now there is little in the preparation of teachers that would help them

[2] For example, William H Burton reported in *The Guidance of Learning Activities* that a child with good standardized test computation scores but unable to solve problems was asked what she did "I know what to do by looking at the examples If there are only two numbers, I subtract If there are lots of numbers, I add. But if there are just two numbers and one is littler than the other, then it is a hard problem I divide to see if they come out even, but if they don't, I multiply " (3d ed , Appleton-Century-Crofts, Inc , New York, 1962



understand such matters. In the next two sections they are treated in more detail as background for the rather frequent reference that is made to them in the remainder of this book.

0.3. APPLICATIONS OF MATHEMATICS AND MATHEMATICAL MODELS

The use of applied mathematics in its relation to a physical problem involves three stages: (1) a dive from the world of reality into the world of mathematics; (2) a swim in the world of mathematics; (3) a climb from the world of mathematics back into the world of reality, carrying a prediction in our teeth.[3]

Starting *thousands* of years ago, man invented numbers, computation, and geometry to help understand and keep track of his world. For the past few *hundreds* of years mathematics has been seen as having "unreasonable effectiveness" in providing equations, formulas, and other mathematical models to help understand real world events related to astronomy and physical sciences. But only in the past few *tens* of years, especially since about 1950, when the first practical electronic computers were built, has mathematics been used extensively in many fields outside the physical sciences. As a result, statements such as these are now commonplace:

[3] J Synge, quoted in the *American Mathematical Monthly,* October 1961, p 799

For economics:

> The applied contributions of mathematical economics cover a wide range of areas. It has helped in the planning and analysis of . . . the measures designed to eliminate recessions and inflations. . . . It has helped to promote efficiency and reduce costs in the selection of portfolios of stocks and bonds, and to the planning of expanded industrial capacities and public transportation networks. In economic theory, it has helped us to investigate more deeply the process of economic growth and the mechanism of business cycles. In these and many other areas, the use of mathematics has become commonplace and has helped to extend the frontiers of research.[4]

For biology:

> There now exists, at least in outline, a systematic mathematical biology which, in the words of one of its pioneers, is "similar in its structure and aims (though not in content) to mathematical physics." . . . Moreover, this mathematical biology has already greatly enriched the biological sciences.[5]

For business:

> The use of mathematical language . . . is already desirable and will soon become inevitable. Without its help the further growth of business with its attendant complexity of organization will be retarded and perhaps halted. In the science of management, as in other sciences, mathematics has become a "condition of progress."[6]

The process by which mathematics becomes useful to workers in these fields is indicated by the quotation at the beginning of this section. The "dive into the world of mathematics" typically results in what is called a "mathematical model" of the real world problem—some bit of mathematics that expresses in abstract terms something about the real situation: a numerical expression, some equations, a geometric diagram, and so on.

[4] W J Baumol, "Mathematics in Economic Analysis," in T L Saaty and F W Weyl, *The Spirit and Uses of the Mathematical Sciences,* McGraw-Hill Book Company, New York, 1969, p 246

[5] A Rosen, "On Mathematics and Biology," also in *The Spirit and Uses of the Mathematical Sciences,* p 204

[6] A Battersby, *Mathematics in Management,* p 11 In this book, whenever the authors make only brief reference to some source, full information about it will be found in Appendix II.

This same process of using abstract mathematical models to express something going on in the world also characterizes everyday uses of mathematics. For example, the counting numbers 1, 2, 3, appear to have been needed, and hence invented, early in human history by virtually every human society that we know about. This "dive into the world of mathematics" was first by way of *spoken* words; written number systems were a later development. Such things as "addition" and "multiplication" of counting numbers were invented to describe something about what happens when sets of things are combined. Fractions may have been invented to express measures.

That is, for centuries man has applied numbers in ways not very different in basic spirit and method from the modern use of mathematical models to solve complicated problems in business, science, government, or social sciences. This method involves examining and simplifying the real world situation until it is possible to describe what you want to know about it in some mathematical terms. One works strictly with the mathematics for a "solution" or answer, then checks this answer to the mathematical problem back in the original real world problem. If the mathematical solution does not fit the real world situation, then one goes through another such round (and perhaps another and another), until an appropriate fit is achieved between some bit of mathematics and some part of the real world.

0.4. EMBODIMENTS OR CONCRETIZATIONS

Many of the activities in this book involve starting the study of a given mathematical concept by dealing with "embodiments" of the concept in actions on real world things. The process is that indicated earlier: Start with a mathematical abstraction and seek to understand it better by acting it out with such things as counters, sticks with graduated lengths, a board with a grid of nails on it, cardboard, string, or whatever works.

Embodiments of mathematics are not mathematics. The ultimate goal is still to be able to understand and work with mathematical symbols and structures. But working with embodiments has proved to be very helpful to children and to adults in achieving this goal. Mathematical "knowledge" for too many people consists of a verbal and symbolic superstructure, much of which lacks meaning. Naturally they find it difficult to use such "knowledge" or teach it to others. We have found that even people with considerable and excellent mathematical training often gain new insights into both mathematics and the teaching of mathematics by putting themselves through the exercise of using concrete materials to act out mathematical abstractions.

For those who are going to teach children, using embodiments of mathematics concepts is especially useful. It is almost certainly true that many children (especially young children) learn mathematics best and come to feel better about it by operating on real objects. For teachers who wish to understand how this sort of learning takes place in children, there is no substitute for going through a similar experience themselves.

One caution is in order, however. This is a course for adults, and over and over again it moves from operations on concrete objects to consideration of patterns of results from these operations, and from there to abstract structures that describe these patterns. The process is similar to what might happen over several years with children, but here it is compressed into a fraction of the time required with children and calls on mature patterns of thinking that may not even be available in young children. Hence, it would almost certainly be hazardous to use the exercises and worksheets in this book directly with children. Still, the experiences outlined here are suggestive of what might be done in teaching mathematics to children, and with suitable adaptation and elaboration many of the "embodiments" and worksheets can become useful teaching resources.

One other remark: Beginning number work in the early school grades does sometimes include concretization by use of counters and counting. But *measure* is the source of numbers to be processed at least as often as is counting. However, the measure motivation for the use of numbers is frequently neglected in instruction of children. Therefore, whenever appropriate, this book has used measure situations as a source of number ideas. Embodiments that reflect measure considerations are given as much prominence as embodiments that reflect counting processes.

0.5. ACTIVITIES AND LABORATORY WORK

The "activities" approach that is used in this book emphasizes that mathematics is as much a process (something one does) as it is a product (something one possesses). Through the use of problem-solving situations, we hope to encourage you to estimate reasonable answers, generate hypotheses, formulate models, investigate regularities, and test and modify hypotheses. Because mathematics is created by people and exists only in their minds, it must be recreated by every person who learns it; this means that some freedom must be allowed for mathematical inquiry. However, allowing freedom for mathematical inquiry does not mean giving no guidance. Indeed, this book provides considerable guidance by the nature of the problem situations that are presented, and the nature of the concrete materials that are used.

We hope that some of the activities in this book will make you want to get involved in *doing* mathematics. To create a balance between giving you a chance to think your own thoughts and forcing you to communicate with (i.e., learn from and teach) other students, many of the activities that we present will encourage you to interact with other students—check your findings with theirs, ask for or give help, talk about any new insights, and so on.

Wherever it is not mathematically misleading, new topics will be presented through the use of simple, concise explanations, together with concrete illustrations and everyday examples. The procedure in most sections will be to introduce concepts in an intuitive fashion before the ideas are gradually refined and formalized.

If advance preparation or special materials are required for a lesson, you can know this ahead of time by looking at the overview to the lesson. Overviews describe what each lesson is generally about and describe the materials that will be needed. If such materials are not available, Appendix I tells how the equipment can be made for inexpensive materials.

Although the use of laboratory materials is rather unusual in mathematics classes, it is common practice in many other subject areas. We believe that the "homemade" materials that are suggested here are the kind that will be useful to you in many teaching and learning situations. Our students have found that by the end of the course they not only have a firm understanding of the major ideas, but they have also accumulated a personal collection of instructional materials that they feel comfortable about using in a wide range of situations.

A criticism that instructors and students sometimes have about activities programs is that concrete problem-solving sessions take time, perhaps at the expense of covering more content. We don't believe that good laboratory activities result in time wasted, but we realize that an activities program requires that efficient use be made of homework assignments. Consequently, some sections of the book are especially suited for in-class laboratory sessions, some for in-class discussions, some for outside reading research assignments, and some for homework or research assignments. These homework assignments frequently ask for thought and analysis with respect to the concepts introduced by the laboratory experience. All these avenues to learning will need to be used, with each of you providing for yourself the links among them.

We have found that an activities-oriented course works best with at least 2-hour class sessions in a room with tables or with movable desks that give space to use materials and opportunities for some work in small groups. But college schedules and room assignments often get in the way of setting up classes in this way. With some ingenuity it is possible to adapt the course to fit whatever scheduling and room conditions are imposed, perhaps even with much of the laboratory activity taking place outside of regularly scheduled "lecture" hours. The "lecture" hours can then be times for discussion and organization of the findings from the outside activities. In such cases there is a much greater burden on you, the student, to take responsibility for your own learning, we hope, by considerable interaction with your fellow students.

0.6. ORGANIZATION OF THIS BOOK

The book is organized in three parts. Part A begins with a thorough treatment of whole numbers, especially of the underlying meaning of the various operations with whole numbers. Here, as elsewhere, there is a stress on applications of numbers—why we need them and how they are used. A series of "extensions" to other number systems then takes place. In each such extension the first question is always, "Why bother? Of what use would the extended system be?" That answered, an extended set of numbers is constructed, the possible new meanings of such relations as "equal" and "less than" are explored, and new definitions and properties of the standard operations are developed, including the various uses to which such operations might be put.

Part B begins with a much more complete investigation of algorithmic processes than is usual in this kind of book. These processes can provide examples of the practical use of "properties" and increased understanding of some of the consequences of various characteristics of our system of numeration. We stress such matters because computational routines and notation are not trivial matters in using mathematics and because, for better or worse, such concerns constitute a very large part of the elementary school curriculum with which teachers must cope. This part of the book also includes a discussion of decimals motivated by the metric system of measurement. Also, the extension of rational numbers to the real numbers gets brief, but honest, treatment.

Part C is perhaps the most self-consciously "mathematical" of the three parts, but its reliance on concrete embodiments is as great as that in the other two parts. It deals in a general way with underlying algebraic concepts (e.g., sets and logic, relations, operations) that occurred throughout Parts A and B.

Since the three series of units are relatively independent, the units in Part A do not have to be completed before Parts B and C are begun. The units are independent enough to allow repeated fresh starts to students who are having difficulties and to allow instructors to modify the sequential presentation of units without destroying the logical themes that run throughout the book. In this way, we hope that the program can be easily adapted to the needs of individual colleges and universities and that instructors will feel free to organize their courses to fit the needs of their students.

Overviews inserted at the beginning of many sections attempt to improve the readability of the book by helping you distinguish the important ideas from the relatively less important ideas as you read through each unit. The overviews also indicate what materials are used in the lesson, including some that may need to be made outside of class prior to the lesson.

In the text itself, there are often footnotes that give further information or give answers to questions so that you can check your work. "Answer" footnotes are printed in blue. Try to resist the temptation to look at "the answer" before really coping with the problem yourself. Problems that are "optional," in the sense of being either relatively less important or more difficult than normal, are marked with an asterisk.

The primary intention of this book is to teach good mathematics. We have included brief "pedagogical remarks" at the end of most units to help you make the transition from learning mathematics yourself to teaching it to children. You should also have a close look at one or more textbook series actually used in schools and read some of the books on the theme of how children learn mathematics that are listed in Appendix II.

Rods and chips and a few other materials are needed in many of the units. Some suggestions about acquiring and managing such materials with a minimum of fuss are in Appendix I.

About one-fourth of this book is a do-it-yourself project. There are many places in which you are asked to write in the results of your investigations or the conclusions that you and your classmates have arrived at as the result of some activity. When you have finished the book by actually filling in these blanks in the lessons, you should have a reference book of considerable value, especially if you are preparing yourself for the exciting task of teaching mathematics to children.

contents