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Bruce A. Reed Cláudia L. Sales
Editors

Recent Advances in Algorithms and Combinatorics

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Recent Advances in Algorithms and Combinatorics

With 52 Illustrations

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Preface

Combinatorics is one of the fastest growing fields of mathematics. In large measure this is because many practical problems can be modeled and then efficiently solved using combinatorial theory. This real world motivation for studying algorithmic combinatorics has led not only to the development of many software packages but also to some beautiful mathematics which has no direct application to applied problems. In this volume we highlight some exciting recent developments in algorithmic combinatorics.

Most practical applications of algorithmic combinatorics would be impossible without the use of the computer. As computers become ever more powerful, more and more applications become possible. Computational biology is one example of a relatively new field in which algorithmic combinatorics plays a key role. The chapter by Sagot and Wakabayashi in this volume discusses how combinatorial tools can be used to search for patterns in DNA and protein sequences.

The information technology revolution has not only allowed for the resolution of practical problems using combinatorial techniques, it has also been the source of many new combinatorial problems. One example is radio channel assignment. In this problem we have a number of transmitters each of which must handle a number of calls. Each call must be assigned a frequency in such a way that interference is avoided (thus calls handled by the same transmitter are assigned different frequencies as are calls handled by transmitters which are *near* each other). The explosive growth in the use of the frequency spectrum due to, e.g., mobile telephone networks, has made it a very valuable resource. Indeed spectrum licenses were sold for billions of dollars in recent actions. So, efficiently assigning radio channels is of great importance. In his chapter in this volume, McDiarmid describes how to model radio channel assignment as a graph colouring problem and surveys the results that have been obtained using this approach.

Using graph colouring models to aid in studying how to direct the flow of information through transmission channels is not new. Shannon defined the zero-error capacity of a noisy (memoryless) channel as the maximum number of bits per symbol which could be sent through the channel whilst avoiding the introduction of errors. In 1961, Berge noted that determining the Shannon capacity of a channel could be modeled as a graph theory problem. In this context, he defined the class of perfect graphs, and noted that for certain channels, the Shannon capacity was simply the chromatic number of the associated perfect graph.

Berge's work motivated considerable research into efficient algorithms for colouring perfect graphs. This problem was finally resolved by Grötschel, Lovász, and Schrijver in 1981 using the (then) recently developed ellipsoid method. They modelled the problem as a semi-definite program(SDP) and then showed how the ellipsoid method could be used to solve this specific SDP. They later showed that in fact the ellipsoid method could be used to solve (actually approximately solve to arbitrary precision) a wide variety of SDP. It turned out that many combinatorial problems can be solved, at least approximately, by solving a related SDP. The most well-known example is the Goemans-Williamson algorithm to approximate Max-Cut. We are fortunate to have a chapter by Lovász in the volume which presents the basic theory of semi-definite programming and surveys its role in combinatorial optimization.

The ellipsoid method is a heavy piece of artillery, and researchers still hope to develop a combinatorial algorithm for colouring perfect graphs, which does not require its use. In his chapter, Maffray surveys some of the approaches with which this problem has been attacked. Many of the techniques for graph colouring he discusses are of interest in their own right and have applications to other graph colouring problems.

Although, the SDP artillery developed by Grötschel, Lovász, and Schrijver is incredibly powerful and beautiful, solving a graph theory problem using this artillery generally yields little insight as to how the optimal solution is determined by the graph's structure. Algorithms developed using decomposition theory, in contrast, often provide such information. Typically when using this paradigm, we decompose the graph into pieces which are easy to deal with, in such a way that it is easy to paste the solutions on the pieces together to obtain a global solution.

The first chapter in this volume is a beautifully written overview of one very important theory of this type. The theory was developed by Lovász to characterize the matching lattice (the matching lattice of a graph is the set of vectors indexed by its edges generated by the incidence vectors of perfect matchings). It was further refined by the authors of this chapter Carvalho, Lucchesi, and Murty.

Another very important theory of this type, that of tree width and tree decompositions, was developed by Robertson and Seymour as part of their seminal work characterizing graphs without a given fixed graph as a minor. In his chapter, Reed discusses the algorithmic aspects of tree decompositions, and mentions some applications to the theory of such diverse fields as databases, code optimization, and bioinformatics.

The third decomposition theorem discussed in this book is Szemerédi's regularity lemma. Roughly speaking, this result tells us that any large graph can be decomposed into edge disjoint random-looking bipartite graphs.

Thus the pieces in this decomposition are easy to deal with because they have many of the properties of random graphs. The basics of this theory is presented in the chapter of Kohayakawa and Rödl, who also survey some of its algorithmic applications. There are many equivalent definitons of what it means to be random looking, or formally *quasi-random*. In their chapter, Kohayakawa and Rödl present a new definition and show that this allows for more efficient algorithms to test this property. This important new result leads to similar efficiency gains in many of the algorithms developed using this theory.

Probability plays a different role in Steger's chapter on approximation algorithm. Recently, a link has been developed between the length of time needed to solve a problem using a deterministic algorithm and the number of bits needed to solve it using a random algorithm (with a given time complexity). This link has allowed researchers to show that many NP-complete optimization problems cannot be approximated unless $P = NP$. Steger's chapter provides an overview of this and other developments in this important field.

One use of graphs as models is to capture the intersection properties of various structures. In this context, the vertices correspond to the structures and two are joined by an edge if they intersect. For example, we can crudely model radio channel assignment in this way, To do so, we think of the vertices as discs around the transmitters representing the area which their broadcast signal covers, and join two vertices by an edge if these discs intersect. Then transmitters with an edge between them must use different frequencies.

Szwarcfiter's chapter considers a self-referential use of graphs of this kind. Here, the vertices of a graph G correspond to the cliques of some other graph H. We join two vertices of G by an edge if the corresponding cliques of H intersect in a vertex. We say that G is the clique graph of H. Szwarcfiter discusses various results on the class of clique graphs.

We have tried to point out some of the intersections between the topics treated in the various chapters of this work. The reader will stumble upon many more as he makes his way through it. More importantly, he will discover that each chapter can be appreciated and enjoyed in its own right.

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Contents

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The Matching Lattice

M.H. de Carvalho[1]
C.L. Lucchesi[2]
U.S.R. Murty[3]

1.1 Perfect Matchings

A set M of edges of a graph G is a *matching* of G if each vertex of G is incident with at most one edge of M and a *perfect matching* of G if each vertex of G is incident with precisely one edge of M.

The fundamental problem of characterizing graphs that admit a perfect matching was settled first for bipartite graphs by Hall in 1935, and more generally, for all graphs by Tutte in 1947. The two well-known theorems which provide these characterizations are given below:

Theorem 1.1.1 (Hall, 1935 [10]) *A graph G, with bipartition $\{A, B\}$, has a perfect matching if and only if $|A| = |B|$ and, $|I(G - S)| \leq |S|$, for each subset S of B, where $I(G - S)$ denotes the set of isolated vertices of $G - S$.*

Hall's Theorem is usually stated differently, but this version, in terms of isolated vertices, has a strong similarity with the statement of Tutte's Theorem, stated below.

Theorem 1.1.2 (Tutte, 1947 [24]) *A graph G has a perfect matching if and only if $|\mathcal{O}(G - S)| \leq |S|$, for each subset S of $V(G)$, where $\mathcal{O}(G - S)$ denotes the set of odd components of $G - S$.*

We denote by $\mathcal{M}(G)$ the set of all perfect matchings of G, or simply by $\mathcal{M}$, if G is understood. Tutte's Theorem gives a characterization of graphs for which $\mathcal{M}$ is nonempty. There are a number of interesting problems in

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graph theory which are concerned with properties of graphs which have a perfect matching. A natural setting for the study of these problems is the theory of matching covered graphs.

1.2 Matching Covered Graphs

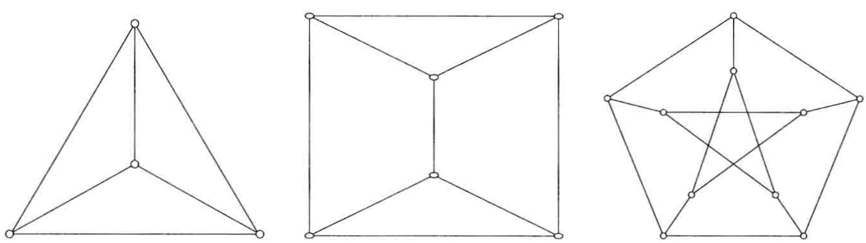
An edge of a graph G is *admissible* if it lies in some perfect matching of G. A *matching covered* graph is a connected graph each edge of which is admissible. Using Hall's and Tutte's Theorems, it is easy to derive the following characterizations of matching covered graphs.

Theorem 1.2.1 *Let G be a bipartite graph with bipartition $\{A, B\}$ that has a perfect matching. Graph G is matching covered if and only if for each nontrivial partition (A', A'') of A and each partition (B', B'') of B such that $|A'| = |B'|$, at least one edge of G joins some vertex of A' to some vertex of B''.*

Theorem 1.2.2 *A connected graph G is matching covered if and only if for each subset S of $V(G)$ the inequality $|\mathcal{O}(G - S)| \leq |S|$ holds, with equality only if set S is independent.*

(A set of vertices S of a graph G is *independent* (or *stable*) if the subgraph $G[S]$ of G spanned by S is free of edges.)

It can be shown that every matching covered graph is 2-connected. Using Theorem 1.2.2, it is easy to show that every 2-edge-connected cubic graph is matching covered. There are three cubic graphs which play particularly important roles in this theory. They are the complete graph K_4, the triangular prism $\overline{C_6}$, and the Petersen graph P (see Figure 1.1).



(a) K_4 (b) $\overline{C_6}$ (c) P

Figure 1.1. Three important cubic graphs.

For a history of the matching covered graphs, see Lovász and Plummer [13]. The most important source for this work is Lovász [12]. Murty [14] is also a very useful reference.

1.3 The Matching Lattice

Let G be a matching covered graph. For each set F of edges of G, we denote by χ^F the *incidence vector* of F in $\mathbb{R}^E$, that is, the vector w of 0's and 1's such that a coordinate $w(e)$ corresponding to edge e of G is equal to 1 if and only if edge e lies in set F. For any integer k, we denote by $\underline{\mathbf{k}}$ the vector of $\mathbb{R}^E$ whose coordinates are all equal to k. For every set F of edges of G and any vector w in $\mathbb{R}^E$, $w(F)$ denotes the scalar product of w and χ^F, that is, $w(F) = \sum_{e \in F} w(e)$.

The linear space generated by the incidence vectors of perfect matchings in G is the *matching space* of G and is denoted by $Lin(G)$:

$$Lin(G) := \{w \in \mathbb{R}^E : w = \sum_{M \in \mathcal{M}} \alpha_M \chi^M, \alpha_M \in \mathbb{R}\}.$$

Likewise, the lattice generated by the incidence vectors of perfect matchings in G is the *matching lattice* of G and is denoted $Lat(G)$:

$$Lat(G) := \{w \in \mathbb{Z}^E : w = \sum_{M \in \mathcal{M}} \alpha_M \chi^M, \alpha_M \in \mathbb{Z}\}.$$

We may restrict the set of coefficients used in the linear combinations to that of the set of nonnegative rationals $\mathbb{Q}_{\geq 0}$ or integers $\mathbb{Z}_{\geq 0}$, thereby obtaining the *rational cone* or *integer cone* spanned by the incidence vectors of matchings in $\mathcal{M}(G)$, denoted $Rat.Con(G)$ or $Int.Con(G)$, respectively:

$$\begin{aligned} Rat.Con(G) &:= \{w \in \mathbb{Q}^E_{\geq 0} : w = \sum_{M \in \mathcal{M}} \alpha_M \chi^M, \alpha_M \in \mathbb{Q}_{\geq 0}\}, \\ Int.Con(G) &:= \{w \in \mathbb{Z}^E_{\geq 0} : w = \sum_{M \in \mathcal{M}} \alpha_M \chi^M, \alpha_M \in \mathbb{Z}_{\geq 0}\}. \end{aligned}$$

Tait showed that the Four-Colour Conjecture is equivalent to the following assertion:

Conjecture 1.3.1 ([23]) *Every 2-connected cubic planar graph is 3-edge colourable.*

If G is a k-regular graph, then G is k-edge colourable if and only if $\underline{\mathbf{1}}$ can be expressed as a sum of incidence vectors of k perfect matchings of G. In other words, G is k-edge colourable if and only if $\underline{\mathbf{1}}$ lies in $Int.Con(G)$. Tutte made the following generalization of the Four-Colour Conjecture:

Conjecture 1.3.2 ([25]) *Every 2-connected cubic graph free of Petersen minors is 3-edge colourable.*

A *minor* of a graph G is any graph that may be obtained from a subgraph of G by edge contractions. Clearly, $Int.Con \subseteq Lat$. This observation led Seymour to study the matching lattice of certain cubic graphs:

Theorem 1.3.3 ([21]) *For every 2-edge-connected cubic graph G, if G does not contain the Petersen graph as a minor, then $\underline{1} \in Lat(G)$.*

We note that Theorem 1.3.3 may be regarded as a proof of a relaxation of Conjecture 1.3.2.

Robertson, Sanders, Seymour and Thomas gave a new proof of the Four-Colour Theorem in [15]. Extending the techniques they developed in that proof, they gave a proof of Conjecture 1.3.2 [17, 19, 18, 16]. Thus, vector $\underline{1}$ lies in the integer cone of any cubic 2-edge-connected planar graph.

Seymour also proved the following assertion:

Theorem 1.3.4 ([21]) *For every 2-edge-connected cubic graph G, vector $\underline{2}$ lies in $Lat(G)$.*

Theorem 1.3.4 may be regarded as a proof of a relaxation of a conjecture due to Fulkerson and Berge [8]:

Conjecture 1.3.5 *For every 2-edge-connected cubic graph G, vector $\underline{2}$ lies in $Int.Con(G)$.*

Lovász, [12], generalized Theorems 1.3.3 and 1.3.4 of Seymour by establishing a complete characterization of the matching lattice of any matching covered graph. More specifically, for any matching covered graph G, and any w in $\mathbb{Z}^E$, Lovász determined the necessary and sufficient conditions for w to be in $Lat(G)$.

We begin with an obvious necessary condition. For any subset S of V, $C = \nabla_G(S)$ (or simply $C = \nabla(S)$) denotes the (edge-) cut of G with S and $\overline{S} = V - S$ as its *shores*; in other words, $\nabla(S)$ is the set of all edges of G which have precisely one end in S. Clearly, for any perfect matching M and any vertex v, $\chi^M(\nabla(v)) = 1$. Therefore, if $w = \sum_{M \in \mathcal{M}} \alpha_M \chi^M$, then, for any vertex v, $w(\nabla(v)) = \sum_{M \in \mathcal{M}} \alpha_M$. A vector w in $\mathbb{R}^E$ is *regular* over a set $\mathcal{C}$ of cuts of G if $w(C) = w(D)$ for any two cuts C and D in $\mathcal{C}$. Vector w is *regular* if it is regular over the set of all stars $\{\nabla(v) : v \in V\}$. In view of the above observation, we have:

Lemma 1.3.6 *For every matching covered graph G, if a vector w lies in $Lat(G)$ then w is regular.*

For matching covered bipartite graphs, the regularity of a vector is also sufficient:

Lemma 1.3.7 *For every bipartite matching covered graph G, a vector in $\mathbb{Z}^E$ lies in $Lat(G)$ if and only if it is regular.*

However, in general, the regularity condition is not sufficient for an integral vector w to belong to $Lat(G)$. For the Petersen graph, $\underline{\mathbf{1}}$ satisfies the regularity condition, but it is not in the matching lattice[4].

In the study of the matching lattice, one makes use of two types of decompositions of matching covered graphs. These decompositions are tight cut decompositions and ear decompositions. We will consider both in the next sections.

1.4 Tight Cut Decompositions

The first type of decomposition of matching covered graphs, known as the *tight cut decomposition*, was introduced by Lovász in [12]. In this section, we describe this procedure and explain its relevance to the study of the matching lattice.

1.4.1 Tight Cuts

Let G be a matching covered graph. A cut C of G is *tight* in G if every perfect matching of G has precisely one edge in C.

Let S be a shore of a cut C of a matching covered graph G. Then, the graph obtained from G by contracting $\overline{S}$ to a single vertex $\overline{s}$ is denoted by $G\{S;\overline{s}\}$ and the graph obtained from G by contracting S to a single vertex s is denoted by $G\{\overline{S};s\}$. We shall refer to these two graphs $G\{S;\overline{s}\}$ and $G\{\overline{S};s\}$ as the *C-contractions* of G. If the names of the new vertices in the C-contractions are irrelevant, we shall simply denote them by $G\{S\}$ and $G\{\overline{S}\}$. Observe that this notation is similar to the notation $G[S]$ used for the subgraph of G induced by S; $G\{S;\overline{s}\}$ is the subgraph induced by S, together with a new vertex $\overline{s}$ such that each edge in $\nabla_G(S)$ joins its end in S to the vertex $\overline{s}$.

Lemma 1.4.1 ([12]) *Let G be a matching covered graph, and let $C = \nabla(X)$ be a tight cut of G. Then the two C-contractions $G_1 = G\{X\}$ and $G_2 = G\{\overline{X}\}$ obtained by contracting the two shores $\overline{X}$ and X, respectively, are also matching covered.*

[4]This follows from the fact that, for any pentagon C of the Petersen graph, $\underline{\mathbf{1}}(C) = 5 \equiv 1\ (mod\ 2)$, whereas $\chi^M(C) \equiv 0\ (mod\ 2)$, for any perfect matching M.

For the Petersen graph, a vector w in Z^E is in the matching lattice if and only if it is regular and, for any pentagon Q, $w(E(Q))$ is even.