GAME THEORY

Second Edition

Guillermo Owen

GAME THEORY

Second Edition

Guillermo Owen

Department of Mathematics
Naval Postgraduate School
Monterey, California

1982

ACADEMIC PRESS
A Subsidiary of Harcourt Brace Jovanovich, Publishers

New York London
Paris San Diego San Francisco São Paulo Sydney Tokyo Toronto

ACADEMIC PRESS, INC.
111 Fifth Avenue, New York, New York 10003

United Kingdom Edition published by
ACADEMIC PRESS, INC. (LONDON) LTD.
24/28 Oval Road, London NW1 7DX

Library of Congress Cataloging in Publication Data

Owen, Guillermo.
Game theory. Second Edition

Bibliography: p.
Includes index.
1. Game theory. I. Title.
QA269.09 1982 519.3 82-4014
ISBN 0-12-531150-8 AACR2

PRINTED IN THE UNITED STATES OF AMERICA

82 83 84 85 9 8 7 6 5 4 3 2 1

PREFACE

It has been a dozen years since the first edition of this book saw print; longer still since that manuscript was first developed. Game theory being a dynamic field, it is not surprising that so much of the book (in fact, practically the entire part dealing with *n*-person games) has required partial or even complete rewriting.

Through the decade of the seventies, many new developments in the theory of games saw light. Some of these were merely expansions of concepts—the nucleolus, nonatomic games, games without side payments—that had been previously introduced. In other cases, totally new concepts—many of these dealing with the problem of information, of which more below—have been developed.

An obvious question is, Why have I introduced none of these new concepts (the nucleolus excepted)? The answer would be that I have, in writing this book, at no time attempted to be comprehensive. It has rather been my desire to give the reader an overview of the mathematical theory of games, sufficiently complete to enable him or her to understand the literature—all this while meeting certain standards for brevity. Under the circumstances, certain topics had to be omitted.

To take a case in point, we might consider the omission of games with incomplete information (not to be confused with imperfect information which is discussed to some extent in Chapter I). One of the fundamental assumptions made throughout this book deals with the *principle of complete information*. This principle says, more or less, that players in a game are, prior to the beginning of play, given the extensive form of the game with all that this implies. In effect, this means that players are at least aware of the legal moves at each moment, of the probabilistic distributions involved, and of the utility which the various outcomes represent, both for themselves and for their opponents.

Such an assumption is probably valid for parlor games; in poker, e.g., it is probably valid to assume that players know the relative ranking of hands and understand the mechanisms of betting. (It is probably incorrect to assume that they know the probabilities involved, but this is normally due to the mathematical complexities of the situation rather than to a lack of information.) For a real-life situation, however, the assumption seems somewhat unrealistic. In particular, the problem of determining other players' capabilities and utility functions seems extremely difficult—all the more so as there is frequently an interest in misrepresentation.

In fact, a substantial amount of work has been done along the lines of weakening—or entirely dispensing with—this assumption. The work is valuable, both from a mathematical and from an applied standpoint. I have nevertheless chosen not to discuss that work in the present volume. My excuse for this is that the work is mathematically quite complex, that it is not necessary to the development of other topics in the book, and that someone who has read this book should be quite capable of understanding the literature in this field. The question is whether this is a valid justification; in all candor I realize that others may disagree with me on this matter.

To discuss the changes which have been made: I have, more or less, taken each of the last three chapters (VIII–X) in the first edition and split it into two parts. I did this because I felt the amount of research along each of the six lines concerned (Chapters VIII–XIII of the present edition) warranted a separate chapter in each case.

On the other hand, certain of the topics that appeared in the first edition—ψ-stability and games in partition function form—have been dropped. I have done this because, in my opinion, not enough work has been done along these lines since 1968 to warrant their continued inclusion in the book. This will no doubt upset some of my colleagues; I trust they will, however, understand the impracticality of an encyclopedic work.

As ever, thanks are in order to certain people and institutions. Both Michael Maschler and Lloyd Shapley have indirectly contributed to this book by the stimulating discussions that I have had with them. Mrs. Velva Power typed all of the new sections of the book. Finally I appreciate the moral support and encouragement given me by my colleagues at the United States Naval Postgraduate School during the last few months of manuscript preparation.

CONTENTS

IV. INFINITE GAMES

V. MULTISTAGE GAMES

VI. UTILITY THEORY

VII. TWO-PERSON GENERAL-SUM GAMES

VIII. *n*-PERSON GAMES

APPENDIX

Chapter I

DEFINITION OF A GAME

I.1 General Notions

The general idea of a game is that with which we are familiar in the context of parlor games. Starting from a given point, there is a sequence of personal moves, at each of which one of the players chooses from among several possibilities; interspersed among these there may also be chance, or random, moves such as throwing a die or shuffling a deck of cards.

Examples of this type of game are chess, in which there are no chance moves (except for the determination of who shall play first), bridge, in which chance plays a much greater part, but in which skill is still important, and roulette, which is entirely a game of chance in which skill plays no part.

The examples of bridge and chess help to point out another important element of a game. In fact, in a chess game each player knows every move that has been made so far, while in bridge a player's knowledge is usually very imperfect. Thus, in some games, a player is unable to determine which of several possible moves has actually been made, either by an opposing player, or by chance. The practical result of this is that, when a player makes a move, he does not know the exact position of the game, and must make his move remembering that there are several possible actual positions.

Finally, at the end of a game, there is normally some payoff to the players (in the form of money, prestige, or satisfaction) which depends on the progress of the game. We may think of this as a function which assigns a payoff to each "terminal position" of the game.

I.2 Games in Extensive Form

In our general idea of a game, therefore, three elements enter: (1) alternation of moves, which can be either personal or random (chance) moves, (2) a possible lack of knowledge, and (3) a payoff function.

We define, first, a *topological tree* or *game tree* as a finite collection of nodes, called *vertices*, connected by lines, called *arcs*, so as to form a connected figure which includes no simple closed curves. Thus it follows that, given any two vertices A and B, there is a unique sequence of arcs and nodes joining A to B.

From this we obtain

I.2.1 Definition. Let Γ be a topological tree with a distinguished vertex A. We say that a vertex C *follows* the vertex B if the sequence of arcs joining A to C passes through B. We say C follows B *immediately* if C follows B and, moreover, there is an arc joining B to C. A vertex X is said to be *terminal* if no vertex follows X.

I.2.2 Definition. By an *n-person game in extensive form* is meant

(α) a topological tree Γ with a distinguished vertex A called the *starting point* of Γ;

(β) a function, called the *payoff function*, which assigns an n-vector to each terminal vertex of Γ;

(γ) a partition of the nonterminal vertices of Γ into $n+1$ sets S_0, $S_1, \ldots, S_n$, called the *player sets*;

(δ) a probability distribution, defined at each vertex of S_0, among the immediate followers of this vertex;

(ε) for each $i = 1, \ldots, n$, a subpartition of S_i into subsets S_i^j, called *information sets*, such that two vertices in the same information set have the same number of immediate followers and no vertex can follow another vertex in the same information set;

(ζ) for each information set S_i^j, an index set I_i^j, together with a 1–1 mapping of the set I_i^j onto the set of immediate followers of each vertex of S_i^j.

The elements of a game are seen here: condition α states that there is a starting point; β gives a payoff function; γ divides the moves into chance moves (S_0) and personal moves which correspond to the n players ($S_i, \ldots, S_n$); δ defines a randomization scheme at each chance move; ε divides a player's moves into "information sets": he knows which information set he is in, but not which vertex of the information set.

I.2.3 Example. In the game of *matching pennies* (Figure I.2.1), player 1 chooses "heads" (H) or "tails" (T). Player 2, not knowing player 1's choice, also chooses "heads" or "tails." If the two choose alike, then player 2 wins a cent from player 1; otherwise, player 1 wins a cent from 2. In the game

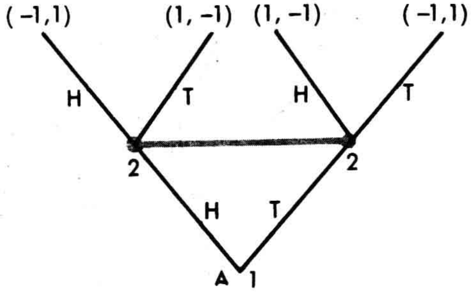

FIGURE I.2.1

tree shown, the vectors at the terminal vertices represent the payoff function; the numbers near the other vertices denote the player to whom the move corresponds. The shaded area encloses moves in the same information set.

I.2.4 Example. The game of pure strategy, or GOPS, is played by giving each of two players an entire suit of cards (thirteen cards). A third suit is shuffled, and the cards of this third suit are then turned up, one by one. Each time one has been turned up, each player turns up one of his cards at will: the one who turns up the larger card "wins" the third card. (If both turn up a card of the same denomination, neither wins.) This continues until the three suits are exhausted. At this point, each player totals the number of spots on the cards he has "won"; the "score" is the difference between what the two players have.

With thirteen-card suits, this game's tree is too large to give here; however, we can give part of the tree of an analogous game using three-card suits (Figure I.2.2).

There is a single chance move, the shuffle, which orders the cards in one of the six possible ways, each having a probability of $\frac{1}{6}$. After this the moves correspond to the two players, I and II, until the game ends. We have drawn parts of the game tree, including the initial point, several branches, and four of the terminal points. The remaining branches are similar to those we have already drawn. With respect to information, we have

I.2.5 Definition. Player i is said to have *perfect information* in Γ if his information sets S_i^j each consist of one element. The game Γ is said to have perfect information if each player has perfect information in Γ.

For example, chess and checkers have perfect information, whereas bridge and poker do not.

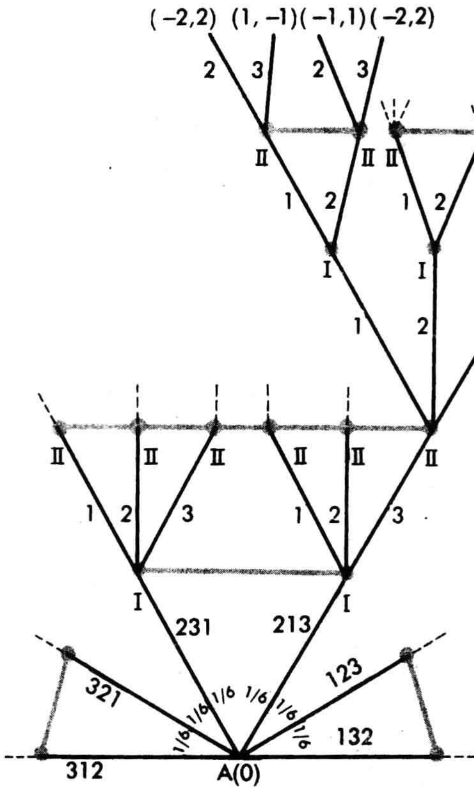

FIGURE I.2.2

I.3 Strategies: The Normal Form

The intuitive meaning of a strategy is that of a plan for playing a game. We may think of a player as saying to himself, "If such and such happens, I'll act in such and such a manner." Thus we have

I.3.1 Definition. By a *strategy* for player i is meant a function which assigns, to each of player i's information sets S_i^j, one of the arcs which follows a representative vertex of S_i^j.

The set of all strategies for player i will be called Σ_i.

In general, we are accustomed to the idea that a player decides his move in a game only a few moves, at best, in advance, and quite usually only at the moment he must make it. In practice this must be so, for in a game such as chess or poker the number of possible moves is so great that no one can plan for every contingency very much in advance. From a purely theoretic point of view, however, we can overlook this practical limitation, and assume that, even before the game starts, each player has already decided what he will do in each case. Thus, we are actually assuming that each player chooses a strategy before the game starts.

Since this is so, it only remains to carry out the chance moves. Moreover, the chance moves may all be combined into a single move, whose result, together with the strategies chosen, determines the outcome of the game.

Actually, what we are interested in, and what the players are interested in, is deciding which one of the strategies is best, from the point of view of maximizing the player's share of the payoff (i.e., player i will want to maximize the ith component of the payoff function). As, however, no one knows, except probabilistically, what the results of the chance moves will be, it becomes natural to take the mathematical expectation of the payoff function, given that the players are using a given n-tuple of strategies. Therefore we shall use the notation

$$\pi(\sigma_1, \sigma_2, \ldots, \sigma_n) = (\pi_1(\sigma_1, \ldots, \sigma_n), \pi_2(\ \ldots\ ), \ldots, \pi_n(\sigma_1, \ldots, \sigma_n))$$

to represent the mathematical expectation of the payoff function, given that player i is using strategy $\sigma_i \in \Sigma_i$.

From this, it becomes possible to tabulate the function $\pi(\sigma_1, \ldots, \sigma_n)$ for all possible values of $\sigma_1, \ldots, \sigma_n$, either in the form of a relation, or by setting up an n-dimensional array of n-vectors. (In case $n = 2$, this reduces to a matrix whose elements are pairs of real numbers.) This n-dimensional array is called the *normal form* of the game Γ.

I.3.2 Example. In the game of matching pennies (see Example I.2.3) each player has the two strategies, "heads" and "tails." The normal form of this game is the matrix

	H	T
H	(−1, 1)	(1, −1)
T	(1, −1)	(−1, 1)

(where each row represents a strategy of player I, and each column a strategy of Player II).

I.3.3 Example. Consider the following game: An integer z is chosen at random, with possible values 1, 2, 3, 4 (each with probability $\frac{1}{4}$). Player I, without knowing the results of this move, chooses an integer x. Player II, knowing neither the result of the chance move nor I's choice, chooses an integer y. The payoff is

$$(|y - z| - |x - z|, |x - z| - |y - z|),$$

i.e., the point is to guess close to z.

In this game each player has four strategies: 1, 2, 3, 4, since other integers are of little use. If, for instance, I chooses 1 and II chooses 3, then the payoff will be $(2, -2)$ with probability $\frac{1}{4}$, $(0,0)$ with probability $\frac{1}{4}$, and $(-2,2)$ with probability $\frac{1}{2}$. The expected payoff, then, is $\pi(1,3) = (-\frac{1}{2},\frac{1}{2})$. Calculating all the values of $\pi(\sigma_1,\sigma_2)$, we obtain

	1	2	3	4
1	$(0,0)$	$(-\frac{1}{2},\frac{1}{2})$	$(-\frac{1}{2},\frac{1}{2})$	$(0,0)$
2	$(\frac{1}{2},-\frac{1}{2})$	$(0,0)$	$(0,0)$	$(\frac{1}{2},-\frac{1}{2})$
3	$(\frac{1}{2},-\frac{1}{2})$	$(0,0)$	$(0,0)$	$(\frac{1}{2},-\frac{1}{2})$
4	$(0,0)$	$(-\frac{1}{2},\frac{1}{2})$	$(-\frac{1}{2},\frac{1}{2})$	$(0,0)$

I.3.4 Definition. A game is said to be finite if its tree contains only finitely many vertices.

Under this definition, most of our parlor games are finite. Chess, for instance, is finite, thanks to the laws which end the game after certain sequences of moves.

It should be seen that, in a finite game, each player has only a finite number of strategies.

I.4 Equilibrium n-Tuples

I.4.1 Definition. Given a game Γ, a strategy n-tuple $(\sigma_1^*,\sigma_2^*, \ldots, \sigma_n^*)$ is said to be *in equilibrium*, or an *equilibrium n-tuple*, if and only if, for any $i = 1, \ldots, n$, and any $\hat{\sigma}_i \in \Sigma_i$,

$$\pi_i(\sigma_1^*, \ldots, \sigma_{i-1}^*, \sigma_i, \sigma_{i+1}^*, \ldots, \sigma_n^*) \leq \pi_i(\sigma_1^*, \ldots, \sigma_n^*).$$

In other words, an n-tuple of strategies is said to be in equilibrium if no player has any positive reason for changing his strategy, assuming that none of the other players is going to change strategies. If, in such a case, each player knows what the others will play, then he has reason to play the strategy which will give such an equilibrium n-tuple, and the game becomes very stable.

I.4.2 Example. In the game with normal form:

	β_1	β_2
α_1	$(2,1)$	$(0,0)$
α_2	$(0,0)$	$(1,2)$

both (α_1, β_1) and (α_2, β_2) are equilibrium pairs.

Unfortunately, not every game has equilibrium n-tuples. As an example, the game of matching pennies (Example I.3.2) has no equilibrium pairs.

In general, if a game has no equilibrium n-tuples, we usually see the several players trying to outguess each other, keeping their strategies secret. This suggests (and it is indeed true) that in games of perfect information, equilibrium n-tuples exist.

To prove this statement, we must study the question of decomposition of a game.

A game Γ will be said to *decompose at a vertex* X if there are no information sets which include vertices from both of (a) X, and all its followers, and (b) the remainder of the game tree. In this case, we can distinguish the subgame, Γ_X, consisting of X, and all its followers, and the quotient game, Γ/X, which consists of all the remaining vertices, plus X. For the quotient game, X will be a terminal vertex; the payoff here can be considered to be Γ_X: i.e., the payoff at this vertex is a play of the subgame Γ_X.

Now, as we have seen, a strategy for i is a function whose domain consists of the information sets of player i. If we decompose a game at X, then we can also decompose the strategy σ into two parts: $\sigma_{|\Gamma/X}$, obtained by restricting σ to information sets in Γ/X, and $\sigma_{|\Gamma_X}$, obtained by restricting σ to Γ_X. Conversely, a strategy for Γ/X and a strategy for Γ_X can be combined in the obvious way to yield a strategy for the larger game Γ.

I.4.3 Theorem. Let Γ decompose at X. For $\sigma_i \in \Sigma_i$, assign to X (considered as a terminal vertex of Γ/X) the payoff

$$\pi_X(\sigma_{1|\Gamma_X}, \sigma_{2|\Gamma_X}, \ldots, \sigma_{n|\Gamma_X}).$$

In this case,

$$\pi(\sigma_1, \ldots, \sigma_n) = \pi_{\Gamma/X}(\sigma_{1|\Gamma/X}, \ldots, \sigma_{n|\Gamma/X}).$$

The proof of this theorem is clear and can be left as an exercise to the reader. Briefly, it is only necessary to verify that, for each possible outcome of the chance moves, the same terminal vertex is eventually reached either in the original or in the decomposed game.

With this, we can prove

I.4.4 Theorem. Let Γ decompose at X, and let $\sigma_i \in \Sigma_i$ be such that (a) $(\sigma_{1|\Gamma_X}, \ldots, \sigma_{n|\Gamma_X})$ is an equilibrium n-tuple for Γ_X, and (b) $(\sigma_{1|\Gamma/X}, \ldots, \sigma_{n|\Gamma/X})$ is an equilibrium n-tuple for Γ/X, with the payoff $\pi(\sigma_{1|\Gamma_X}, \ldots, \sigma_{n|\Gamma_X})$ assigned to the terminal vector X. Then $(\sigma_1, \ldots, \sigma_n)$ is an equilibrium n-tuple for Γ.

Proof: Let $\hat{\sigma}_i \in \Sigma_i$. Because $(\sigma_{1|\Gamma_X}, \ldots, \sigma_{n|\Gamma_X})$ is an equilibrium n-tuple for Γ_X, it follows that

$$\pi_i(\sigma_{1|\Gamma_X}, \ldots, \hat{\sigma}_{i|\Gamma_X}, \ldots, \sigma_{n|\Gamma_X}) \leq \pi_i(\sigma_{1|\Gamma_X}, \ldots, \sigma_{n|\Gamma_X}).$$

On the other hand, by (b) we know that, if we assign the payoff $\pi(\sigma_{1|\Gamma_X}, \ldots, \sigma_{n|\Gamma_X})$ to the vector X, then

$$\pi_i(\sigma_{1|\Gamma/X}, \ldots, \hat{\sigma}_{i|\Gamma/X}, \ldots, \sigma_{n|\Gamma/X}) \leq \pi_i(\sigma_{1|\Gamma/X}, \ldots, \sigma_{n|\Gamma/X}).$$

Now, the payoff (for a given set of strategies) is a weighted average of the payoffs at some of the terminal vertices of a tree. Hence, if the payoff to player i at a given terminal vertex (X, in this case) is decreased, his expected payoff, for any choice of strategies, will either remain equal or be decreased. Thus, applying Theorem I.4.3, we find that

$$\pi_i(\sigma_1, \ldots, \hat{\sigma}_i, \ldots, \sigma_n) \leq \pi_i(\sigma_1, \ldots, \sigma_n),$$

and so $(\sigma_1, \ldots, \sigma_n)$ is an equilibrium n-tuple.

This is all we need to prove

I.4.5 Theorem. Every finite n-person game with complete information has an equilibrium n-tuple of strategies.

Proof: We shall define the length of a game as the largest possible number of edges that can be passed before reaching a terminal vertex, i.e., the largest possible number of moves before the game ends. Clearly a finite game has finite length. The proof is by induction on the length of the game.

If Γ has length 0, the theorem is trivially true. If it has length 1, then at most one player gets to move, and he obtains equilibrium by choosing his best alternative. If Γ has length m, then it decomposes (having complete information) into several subgames of length less than m. By the induction hypothesis, each of these subgames has an equilibrium n-tuple; by Theorem I.4.4 these form an equilibrium n-tuple for Γ.

Problems

1. An infinite game, even with perfect information, need not have an equilibrium n-tuple.

   (a) Consider a two-person game in which the two players alternate, and, at each move, each player chooses one of the two digits 0 and 1. If the digit x_i is chosen at the ith move, then each play of the game