CONTEMPORARY MATHEMATICS

450

Poisson Geometry in Mathematics and Physics

International Conference
June 5–9, 2006
Tokyo, Japan

Giuseppe Dito
Jiang-Hua Lu
Yoshiaki Maeda
Alan Weinstein
Editors

American Mathematical Society

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Foreword

The conference "Poisson 2006: Poisson Geometry in Mathematics and Physics" was held from June 5 through 9, 2006 at the National Olympics Memorial Youth Center in Tokyo. There were about 150 participants, including 25 invited speakers, and 20 presenters at a poster session.

The speakers were chosen by a Scientific Committee of ten members, chaired by Alan Weinstein, while local organization was handled by a separate committee headed by Yoshiaki Maeda and Giuseppe Dito.

The meeting was preceded by a school of about three days, organized by Giuseppe Dito, Yoshiaki Maeda and Alan Weinstein, consisting of a lecture series designed to provide background for the conference talks, as well as invited topical lectures by young participants.

Sponsoring organizations for the conference and school included the Mathematical Society of Japan, the European Mathematical Society, the American Mathematical Society, the Bernoulli Center at EPFL Lausanne, and the 21st Century Center of Excellence (COE) at Keio University. The COE provided the majority of funding, with additional support from the US National Science Foundation.

Poisson 2006 was the fifth in a series of international conferences on Poisson geometry, held every two years. The first, in 1998, took place at the Banach Centre in Warsaw, with subsequent meetings at CIRM in Luminy, IST in Lisbon, and the University of Luxembourg. Further information about all these meetings, as well as the one to be held in 2008 at EPFL in Lausanne, may be found on the Poisson Geometry Home Page at `poissongeometry.org`, which links to the videos of all the talks of the conference Poisson 2006 and principal lectures of the school.

The aim of these meetings has been to bring together mathematicians and mathematical physicists who work in diverse areas and share a common interest in Poisson geometry. With roots in classical mechanics from 200 years ago and the work of Sophus Lie from a century ago, the subject of Poisson geometry crystallized through the work of Kirillov and Lichnerowicz in the 1970's and has been particularly driven by the program of "deformation quantization", in which Poisson structures appear as the first deviation from commutativity in families of associative algebras. Subjects where Poisson geometry plays an essential role include symplectic geometry and topology, deformation theory, representation theory, hamiltonian dynamics, and field theory.

In preparing the program for Poisson 2006, the Scientific Committee made a special effort to include speakers from "outside" areas which were relevant to Poisson geometry and its applications. The program of Poisson 2006 (conference and school) was remarkable for the overlap of topics, some intentional, some fortuitous, between the lectures. Here are some examples:

- Generalized complex stuctures and other geometry on $TM + T^*M$ (Bursztyn, Gualtieri, Meinrenken, Uchino, Yoshimura)
- Stacks and twisting by a three-form (Gomi, Schapira, Tsygan, Van den Bergh)
- Orbifolds and other singular spaces as differentiable stacks (Crainic, Holm, Weinstein, Xu, Zhu)
- Normal forms of Poisson structures in the neighborhood of points and symplectic leaves (Dufour, Fernandes, Ratiu, Zung)
- Deformation of Poisson structures (Ikeda, Zhang)
- Reduction of systems with symmetry (Bursztyn, Cardona, Cattaneo, Holm, Ratiu, Yoshimura)
- Kontsevich formality and its variants (Alekseev, Kontsevich, Merkulov, Park, Tsygan, Van den Bergh, Waldmann)
- Log-canonical coordinates (Gekhtman, Kontsevich, Lu)
- Group-valued momentum maps (Meinrenken, Schaffhauser)
- Strict quantization of spaces via group actions (Rieffel, Voglaire, Waldmann)
- Quantization of canonical transformations via their graphs (Kontsevich, Schapira, Van den Bergh, Weinstein)

The present volume consists of refereed papers by many of the invited speakers at the conference and by the principal lecturers at the school. Papers by presenters at the poster session and by other speakers at the school will be appearing in *Travaux Mathématiques.*

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Quantized Anti de Sitter spaces and non-formal deformation quantizations of symplectic symmetric spaces

Pierre Bieliavsky, Laurent Claessens, Daniel Sternheimer, and Yannick Voglaire

ABSTRACT. We realize quantized anti de Sitter space black holes, building Connes spectral triples, similar to those used for quantized spheres but based on Universal Deformation Quantization Formulas (UDF) obtained from an oscillatory integral kernel on an appropriate symplectic symmetric space. More precisely we first obtain a UDF for Lie subgroups acting on a symplectic symmetric space M in a locally simply transitive manner. Then, observing that a curvature contraction canonically relates anti de Sitter geometry to the geometry of symplectic symmetric spaces, we use that UDF to define what we call Dirac-isospectral noncommutative deformations of the spectral triples of locally anti de Sitter black holes. The study is motivated by physical and cosmological considerations.

1. Introduction

1.1. Physical and cosmological motivations. This paper, of independent interest in itself, can also be seen as a small part in a number of long haul programs developed by many in the past decades, with a variety of motivations. The references that follow are minimal and chosen mostly so as to be a convenient starting point for further reading, that includes the original articles quoted therein.

An obvious fact (almost a century old) is that anti de Sitter (AdS) space-time can be obtained from usual Minkowski space-time, deforming it by allowing a (small) non-zero negative curvature. The Poincaré group symmetry of special relativity is then deformed (in the sense of [**Ger64**]) to the AdS group $SO(2,3)$. In $n+1$ space-time dimensions ($n \geq 2$) the corresponding AdS_n groups are $SO(2,n)$. Interestingly these are the conformal groups of flat (or AdS) n space-times. The *deformation philosophy* [**Fla82**] makes it then natural, in the spirit of deformation quantization [**DS02**], to deform these further [**St05, St07**], i.e. quantizing them, which many are doing for Minkowski space-time.

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Key words and phrases. Non commutative geometry, BTZ-spaces, Deformation quantization, Symplectic symmetric spaces.

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Laurent Claessens is a FRIA-fellow.

These deformations have important consequences. Introducing a small negative curvature ρ permits to consider [**AFFS**] massless particles as composites of more fundamental objects, the Dirac *singletons*, so called because they are associated with unitary irreducible representations (UIR) of $SO(2,3)$, discovered by Dirac in 1963, so poor in states that the weight diagram fits on a single line. These have been called Di and Rac and are in fact massless UIR of the Poincaré group in one space dimension less, uniquely extendible to the corresponding conformal group $SO(2,3)$ (AdS_4/CFT_3 symmetry, a manifestation of 't Hooft's holography). That kinematical fact was made dynamical [**FF88**] in a manner consistent with quantum electrodynamics (QED), the photons being considered as 2-Rac states and the creation and annihilation operators of the naturally confined Rac having unusual commutation relations (a kind of "square root" of the canonical commutation relations for the photon).

Later [**FFS**] this phenomenon has been linked with the (then very recently) observed oscillations of neutrinos (see below, neutrino mixing). Shortly afterwards, making use of flavor symmetry, Frønsdal was able to modify the electroweak model [**Frø00**], obtaining initially massless leptons (see below) that are massified by Yukawa interaction with Higgs particles. (In this model, 5 pairs of Higgs are needed and it predicts the existence of new mesons, parallel to the W and Z of the $U(2)$-invariant electroweak theory, associated with a $U(2)$ flavor symmetry.)

Quantum groups can be viewed [**BGGS**] as an avatar of deformation quantization when dealing with Hopf algebras. Of particular interest here are the quantized AdS groups [**FHT, Sta98**], especially at even root of unity since they have some finite dimensional UIR, a fact generally associated with compact groups and groups of transformations of compact spaces. It is then tempting to consider quantized AdS spaces at even root of unity $q = e^{i\theta}$ as "small black holes" in an ambient Minkowski space that can be obtained as a limit when $\rho q \to -0$. Note that, following e.g. 't Hooft (see e.g. [**Hoo06**] but his approach started around 1980) that some form of communication is possible with quantum black holes by interaction at their surface.

At present, conventional wisdom has it that our universe is made up mostly of "dark energy" (74% according to a recent Wilkinson Microwave Anisotropy Probe, WMAP), then of "dark matter" (22% according to WMAP), and only 4% of "our" ordinary matter, which we can more directly observe. Dark matter is "matter", not directly observed and of unknown composition, that does not emit nor reflect enough electromagnetic radiation to be detected directly, but whose presence can be inferred from gravitational effects on visible matter. According to the Standard Model, dark matter accounts for the vast majority of mass in the observable universe. Dark energy is a hypothetical form of energy that permeates all of space. It is currently the most popular method for explaining recent observations that the universe appears to be expanding at an accelerating rate, as well as accounting for a significant portion of the missing mass in the universe.

The Standard Model of particle physics is a model which incorporates three of the four known fundamental interactions between the elementary particles that make up all matter (the fourth one, weakest but long range, being gravity). It came after the electroweak theory that incorporated QED (electromagnetic interactions) associated with the photon and the so-called weak interactions, associated with the *leptons* that now exist in three generations (flavors): electron, muon and tau, and