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Martin Greiter

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Preface

The immediate advance we communicate with this monograph is the discovery of an exact model for a critical spin chain with arbitrary spin S, which includes the Haldane–Shastry model as the special case $S = \frac{1}{2}$. For $S \geq 1$, we propose that the spinon excitations obey a one-dimensional version of non-Abelian statistics, where the topological degeneracies are encoded in the fractional momentum spacings for the spinons. The model and its properties, however, are not the only, and possibly not even the most important thing one can learn from the analysis we present.

The benefit of science may be that it honors the human spirit, gives pleasure to those who immerse themselves in it, and pragmatically, contributes to the improvement of the human condition in the long term. The purpose of the individual scientific work can hence be either a direct contribution to this improvement, or more often an indirect contribution by making an advance which inspires further advances in a field. When we teach Physics, be it in lectures, books, monographs, or research papers, we usually teach what we understand, but rarely spend much effort on teaching how this understanding was obtained. The first volume of the famed course of theoretical physics by L. D. Landau and E. M. Lifshitz [1], for example, begins by stating the principle of least action, but does nothing to motivate how it was discovered historically or how one could be led to discover it from the study of mechanical systems. This reflects that we teach our students how to apply certain principles, but not how to discover or extract such principles from a given body of observations. The reason for this is not that we are truely content to teach students of physics as if they were students of engineering, but that the creative process in physics is usually erratic and messy, if not plainly embarrassing to those actively involved, and hence extremely difficult to recapture. As with most of what happens in reality, the actual paths of discovery are usually highly unlikely. Since we enjoy the comfort of perceiving actions and events as more likely and sensible, our minds subconsciously filter our memory to this effect.

One of the first topics I immersed myself in after completing my graduate coursework was Laughlin's theory of the fractionally quantized Hall effect [2].

I have never completely moved away from it, as this work testifies, and take enormous delight whenever I recognize quantum Hall physics in other domains of physics. More important than the theory itself, however, was to me to understand and learn from the way R. B. Laughlin actually discovered the wave function. He numerically diagonalized a system of three electrons in a magnetic field in an open plane, and observed that the total canonical angular momentum around the origin jumped by a factor of three (from $3\hbar$ to $9\hbar$) when he implemented a Coulomb interaction between the electrons. At the same time, no lesser scientists than D. Yoshioka, B. I. Halperin, and P. A. Lee [3] had, in an heroic effort, diagonalized up to six electrons with periodic boundary conditions, and concluded that their data were "supportive of the idea that the ground state is not crystalline, but a translationally invariant "liquid."" Their analysis was much more distinguished and scholarly, but unfortunately, did not yield the wave function.

The message I learned from this episode is that it is often beneficial to leave the path of scholarly analysis, and play with the simplest system of which one may hope that it might give away natures thoughts. For the Laughlin series of quantized Hall states, this system consisted of three electrons. I spend most of my scientific life adapting this approach to itinerant antiferromagnets in two dimensions, where I needed to go to twelve lattice sites until I could grasp what nature had in mind. But I am digressing. To complete the story about the discovery of the quantum Hall effect, Laughlin gave a public lecture in Amsterdam within a year of having received the Nobel price. He did not mention how he discovered the state, and at first couldn't recall it when I asked him in public after the lecture. As he was answering other questions, he recalled the answer to mine and weaved it into the answer of another question. During the evening in a cafe, a very famous Russian colleague whom I regard with the utmost respect commented the story of the discovery with the words "But this is stupid!".

Maybe it is. If it is so, however, the independent discoveries of the spin $\frac{1}{2}$ model by F. D. M. Haldane [4] and B. S. Shastry [5] may fall into the same category. Unfortunately, I do not know much about these discoveries. Haldane told me that he first observed striking degeneracies when he looked at the model for $N = 6$ sites numerically, motivated by the fact that the $1/r^2$ exchange is the discrete Fourier transform of $\epsilon(k) = k(k - 2\pi)$ in one dimension. Shastry told me that he discovered it "by doing calculations", which is not overly instructive to future generations. If my discovery of the general model I document in this monograph will be perceived in the spirit of my friends comment, I will at least have made no attempt to evade the charge.

In short, what I document on these pages is not just an exact model, but a precise and reproducible account of how I discovered this model. This reflects my belief that the path of discovery can be as instructive to future generations as the model itself. Of course, the analysis I document does not fully reflect the actual path of discovery, but what would have been the path if my thinking had followed a straight line. It took me about four weeks to obtain all the results and about four months to write this monograph. The reason for this discrepancy is not that my writing proceeds slowly, but that I had left out many intermediate steps when I did the calculation. The actual path of discovery must have been highly unlikely. In any event, it is comforting to me that, now that I have written a scholarly and coherent account of it, there is little need to recall what actually might have happened.

I am deeply grateful to Ronny Thomale for countless discussions and his critical reading of the manuscript, to Burkhard Scharfenberger, Dirk Schuricht, and Stephan Rachel for collaborations on various aspects of quantum spin chains, to Rose Schrempp and the members of the Institute for Theory of Condensed Matter at KIT for providing me with a pleasant and highly stimulating atmosphere, and especially to Peter Wölfle for his continued encouragement and support.

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Karlsruhe, April 2011 Martin Greiter

References

[1] L.D. Landau, E.M. Lifshitz, *Mechanics* (Butterworth-Heinemann, Oxford, 1982)

[2] R.B. Laughlin, Anomalous quantum hall effect: an incompressible quantum fluid with fractionally charged excitations. Phys. Rev. Lett. **50**, 1395 (1983)

[3] D. Yoshioka, B.I. Halperin, P.A. Lee, Ground state of two-dimensional electronsin strong magnetic fields and $\frac{1}{3}$ quantized hall effect. Phys. Rev. Lett. **50**,1219 (1983)

[4] F.D.M. Haldane, Exact Jastrow–Gutzwiller resonant-valence-bond ground state of the spin-$\frac{1}{2}$ antiferr magnetic Heisenberg chain with $1/r^2$ exchange. Phys. Rev. Lett. **60**, 635 (1988)

[5] B.S. Shastry, Exact solution of an $S = \frac{1}{2}$ Heisenberg antiferromagnetic chain with long-ranged interactions. Phys. Rev. Lett. **60**, 639 (1988)

Contents

Chapter 1
Introduction and Summary

Fractional quantization, and in particular fractional statistics [1, 2], in two-dimensional quantum liquids is witnessing a renaissance of interest in present times. The field started more than a quarter of a century ago with the discovery of the fractional quantum Hall effect, which was explained by Laughlin [3] in terms of an incompressible quantum liquid supporting fractionally charged (vortex or) quasiparticle excitations. When formulating a hierarchy of quantized Hall states [4–7] to explain the observation of quantized Hall states at other filling fractions fractions, Halperin [5, 6] noted that these excitations obey fractional statistics, and are hence conceptually similar to the charge-flux tube composites introduced by Wilczek two years earlier [8]. Physically, the fractional statistics manifests itself through fractional quantization of the kinematical relative angular momenta of the anyons.

The interest was renewed a few years later, when Anderson [9] proposed that hole-doped Mott insulators, and in particular the t–J model [10, 11] universally believed to describe the CuO planes in high T_c superconductors [12, 13], can be described in terms of a spin liquid (i.e., a state with strong, local antiferromagnetic correlations but without long range order), which would likewise support fractionally quantized excitations. In this proposal, the excitations are spinons and holons, which carry spin $\frac{1}{2}$ and no charge or no spin and charge $+e$, respectively. The fractional quantum number of the spinon is the spin, which is half integer while the Hilbert space (for the undoped system) is built up of spin flips, which carry spin one. One of the earliest proposals for a spin liquid supporting deconfined spinon and holon excitations is the (Abelian) chiral spin liquid [14–17]. Following up on an idea by D.H. Lee, Kalmeyer and Laughlin [14, 15] proposed that a quantized Hall wave function for bosons could be used to describe the amplitudes for spin-flips on a lattice. The chiral spin liquid state did not turn out to be relevant to CuO superconductivity, but remains one of very few examples of two-dimensional spin liquids with fractional statistics. Other established examples of two-dimensional spin liquids include the resonating valence bond (RVB) phases of the Rokhsar–Kivelson model [18] on the triangular lattice identified by Moessner and Sondhi [19], of the Kitaev model [20], and of the Hubbard model on the honeycomb lattice [21].

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While usually associated with two-dimensional systems, fractional statistics is also possible in one dimension. The paradigm for one-dimensional anyons are the spinon excitations in the Haldane–Shastry model [22, 23], a spin chain model with $S = \frac{1}{2}$ and long-ranged Heisenberg interactions. The ground state can be generated by Gutzwiller projection of half-filled bands of free fermions, and is equivalent to a chiral spin liquid in one dimension. The unique feature of the model is that the spinons are free in the sense that they only interact through their fractional statistics [24, 25]. The half-fermi statistics was originally discovered and formulated through a fractional exclusion or generalized Pauli principle [26], according to which the creation of two spinons reduces the number of single particle states available for further spinons by one. It manifests itself physically through fractional shifts in the spacings between the kinematical momenta of the individual spinons [27–29].

The present renaissance of interest in fractional statistics is due to possible applications of states supporting excitations with non-Abelian statistics [30] to the rapidly evolving field of quantum computation and cryptography [31, 32]. The paradigm for this universality class, is the Pfaffian state introduced by Moore and Read [33] in 1991. The state was proposed to be realized at the experimentally observed fraction $\nu = \frac{5}{2}$ [34] (i.e., at $\nu = \frac{1}{2}$ in the second Landau level) by Wen, Wilczek, and ourselves [35, 36], a proposal which recently received experimental support through the direct measurement of the quasiparticle charge [37, 38]. The Moore–Read state possesses $p + \mathrm{i}p$-wave pairing correlations. The flux quantum of the vortices is one half of the Dirac quantum, which implies a quasiparticle charge of $e/4$. Like the vortices in a p-wave superfluid, these quasiparticles possess Majorana-fermion states [39] at zero energy (i.e., one fermion state per pair of vortices, which can be occupied or unoccupied). A Pfaffian state with $2L$ spatially separated quasiparticle excitations is hence 2^L fold degenerate [40], in accordance with the dimension of the internal space spanned by the zero energy states. While adiabatic interchanges of quasiparticles yield only overall phases in Abelian quantized Hall states, braiding of half-vortices of the Pfaffian state will in general yield non-trivial changes in the occupations of the zero energy states [41, 42], which render the interchanges non-commutative or non-Abelian. In particular, the internal state vector is insensitive to local perturbations—it can only be manipulated through non-local operations like braiding of the vortices or measurements involving two or more vortices simultaneously. For a sufficiently large number of vortices, on the other hand, any unitary transformation in this space can be approximated to arbitrary accuracy through successive braiding operations [43]. These properties together render non-Abelions preeminently suited for applications as protected qubits in quantum computation [30, 32, 44–46]. Non-Abelian anyons are further established in certain other quantum Hall states described by Jack polynomials [47–49] including Read–Rezayi states [50], in the non-Abelian phase of the Kitaev model [20], in the Yao–Kivelson model [51], and in the non-Abelian chiral spin liquid proposed by Thomale and ourselves [52]. In this liquid, the amplitudes for renormalized spin-flips on a lattice with spins $S = 1$ are described by a bosonic Pfaffian state.

The connection between the Haldane–Shastry ground state, the chiral spin liquid, and a bosonic Laughlin state at Landau level filling fraction $\nu = \frac{1}{2}$ suggests that one may consider the non-Abelian chiral spin liquid in one dimension as a ground state for a spin chain with $S = 1$. This state is related to a bosonic Moore–Read state at filling fraction $\nu = 1$. In this monograph, we will introduce and elaborate on this one-dimensional spin liquid state, construct a parent Hamiltonian, and generalize the model to arbitrary spin S. We further propose that the spinon excitations of the states for $S \geq 1$ will obey a novel form of "non-Abelian" statistics, where the internal, protected Hilbert space associated with the statistics is spanned by topological shifts in the spacings of the single spinon momenta when spinons are present.

Most of the book will be devoted to the construction of the model Hamiltonian for spin S. In Chap. 2, we introduce three exact models, and the ground state for the $S = 1$ spin chain for which we wish to construct a parent Hamiltonian. The exact models consist of Hamiltonians, their ground states, and the elementary excitations, which are in some cases exact and in others approximate eigenstates of the Hamiltonian. In Sect. 2.1, we review the Laughlin $\nu = \frac{1}{m}$ state for quantized Hall liquids,

$$\psi_0(z_1, z_2, \ldots, z_M) = \prod_{i<j}^{M} (z_i - z_j)^m \prod_{i=1}^{M} e^{-\frac{1}{4}|z_i|^2}, \tag{1.1}$$

where the z_i's are the coordinates of M electrons in the complex plane, and m is odd for fermions and even for bosons. For $m = 2$, its parent Hamiltonian is given by the kinetic term giving rise to Landau level quantization supplemented by a δ-function potential, which excludes the component with relative angular momentum zero between pairs of bosons. The ground state wave function for a bosonic $m = 2$ Laughlin state is similar to the ground state of the Haldane–Shastry model we review in Sect. 2.2,

$$\psi_0^{\text{HS}}(z_1, z_2, \ldots, z_M) = \prod_{i<i}^{M} (z_i - z_j)^2 \prod_{i=1}^{M} z_i, \tag{1.2}$$

where the z_i's are now coordinates of spin flips for a spin chain with N sites on a unit circle embedded in the complex plane, and $M = \frac{N}{2}$. The Haldane–Shastry Hamiltonian,

$$H^{\text{HS}} = \left(\frac{2\pi}{N}\right)^2 \sum_{\alpha<\beta}^{N} \frac{\boldsymbol{S}_\alpha \boldsymbol{S}_\beta}{|\eta_\alpha - \eta_\beta|^2}, \tag{1.3}$$

where $\eta_\alpha = e^{\mathrm{i}\frac{2\pi}{N}\alpha}$ are the coordinates of the N sites on the unit circle, however, bears no resemblance to the δ-function Hamiltonian for the Laughlin states. We will elaborate in Sect. 3.1 that these models are both physically and mathematically sufficiently different to consider them unrelated. Even the ground state wave functions, when adapted as far as any possible by formulating the bosonic Laughlin state on the sphere and by inserting a quasihole at the south pole, differ due to different Hilbert space normalizations. From a scholarly point of view, there just appears to be no connection.

From a pragmatic point of view, however, we may view both Hamiltonians as devices to obtain the coefficients of the polynomial

$$\prod_{i<i}^{N}(z_i - z_j)^2$$

for particle numbers such that the Hamiltonians can be diagonalized numerically. In fact, Haldane [4] introduced the parent Hamiltonian for the Laughlin state in order to obtain the coefficients of all the configurations of the state vector for $N = 6$, which he could then compare numerically to the exact ground state for Coulomb interactions. This raises the question whether the recipes used by both Hamiltonians for obtaining these coefficients are really different. If one wishes to attribute the results we presented to a discovery, this discovery is that they are not.

When we "derive" the Haldane–Shastry model from the bosonic $m = 2$ Laughlin state and its δ-function parent Hamiltonian in Chap. 3, we really first extract this recipe from the quantum Hall Hamiltonian, and then use it to construct a parent Hamiltonian for the quantum spin chain, which has to be Hermitian, local, and invariant under translations, parity, time reversal, and SU(2) spin rotations. Written in the language of the spin system, the recipe is the condition that the Haldane–Shastry ground state is annihilated by the operator

$$\Omega_\alpha^{\text{HS}} = \sum_{\substack{\beta=1 \\ \beta\neq\alpha}}^{N} \frac{1}{\eta_\alpha - \eta_\beta} S_\alpha^- S_\beta^-, \qquad \Omega_\alpha^{\text{HS}} |\psi_0^{\text{HS}}\rangle = 0 \quad \forall\, \alpha. \tag{1.4}$$

The Haldane–Shastry model has been known for more than two decades, but while Haldane and Shastry independently discovered it, we derive it. Unlike the discoveries, this derivation lends itself to a generalization to higher spins. The construction of exact models of critical spin chains following the line of reasoning we use in our derivation of the Haldane–Shastry model is the subject of this monograph.

In Sect. 2.3, we review the properties of the Moore–Read state [33, 35, 36],

$$\psi_0(z_1, z_2, \ldots, z_N) = \text{Pf}\left(\frac{1}{z_i - z_j}\right) \prod_{i<j}^{N}(z_i - z_j)^m \prod_{i=1}^{N} e^{-\frac{1}{4}|z_i|^2}, \tag{1.5}$$

at Landau level filling fraction $\nu = \frac{1}{m}$, where m is even for fermions and odd for bosons, with emphasis on the non-Abelian statistics of the half-vortex quasiparticle excitations. For $m = 1$, the Pfaffian state is the exact ground state of the kinetic Hamiltonian supplemented by the three-body interaction term [36]

$$V = \sum_{i,j<k}^{N} \delta^{(2)}(z_i - z_j)\delta^{(2)}(z_i - z_k). \tag{1.6}$$

The bosonic $m = 1$ ground state is similar to the ground state wave function of the critical $S = 1$ spin liquid state we introduce in Sect. 2.4,

$$\psi_0^{S=1}(z_1, z_2, \ldots, z_N) = \mathrm{Pf}\left(\frac{1}{z_i - z_j}\right) \prod_{i<j}^{N}(z_i - z_j) \prod_{i=1}^{N} z_i, \tag{1.7}$$

which describes the amplitudes of renormalized spin flips

$$\tilde{S}_\alpha^+ = \frac{S_\alpha^z + 1}{2} S_\alpha^+, \tag{1.8}$$

on sites $\eta_\alpha = e^{\mathrm{i}\frac{2\pi}{N}\alpha}$ on a unit circle embedded in the complex plane. These spin flips act on a vacuum where all the N spins are in the $S^z = -1$ state. In Sect. 2.4.5, we propose that the momentum spacings between the individual spinon excitations of this liquid alternate between being odd multiples of $\frac{\pi}{N}$ and being either even or odd multiples of $\frac{\pi}{N}$. (Since the spacings for bosons or fermions are multiples of $\frac{2\pi}{N}$, an odd multiply of $\frac{\pi}{N}$ corresponds to half-fermion, and an even multiple to boson or fermion statistics.) When we have a choice between even and odd, this choice represents a topological quantum number. The momentum spacings hence span an internal or topological Hilbert space of dimension 2^L when $2L$ spinons are present, as appropriate for Ising anyons. These spacings constitute the analog of the Majorana fermion states in the cores of the half-vortex excitations of the Moore–Read state.

In Chap. 4, we derive a parent Hamiltonian for the $S = 1$ spin liquid state (1.5) from the three-body parent Hamiltonian (1.6) of the Moore–Read state. The steps are similar to those taken for the Haldane–Shastry model, but technically more involved. The defining condition for the state, i.e., the recipe used by the quantum Hall Hamiltonian to specify the coefficients of the polynomial

$$\mathrm{Pf}\left(\frac{1}{z_i - z_j}\right) \prod_{i<j}^{N}(z_i - z_j),$$

is in the language of the $S = 1$ spin model given by

$$\Omega_\alpha^{S=1} = \sum_{\substack{\beta=1 \\ \beta\neq\alpha}}^{N} \frac{1}{\eta_\alpha - \eta_\beta} (S_\alpha^-)^2 S_\beta^-, \qquad \Omega_\alpha^{S=1} \left|\psi_0^{S=1}\right\rangle = 0 \quad \forall\, \alpha. \tag{1.9}$$

As an aside, we also find that the state is annihilated by the operator

$$\Xi_\alpha = \sum_{\substack{\beta,\gamma=1 \\ \beta,\gamma\neq\alpha}}^{N} \frac{S_\alpha^- S_\beta^- S_\gamma^-}{(\eta_\alpha - \eta_\beta)(\eta_\alpha - \eta_\gamma)} - \sum_{\substack{\beta=1 \\ \beta\neq\alpha}}^{N} \frac{(S_\alpha^-)^2 S_\beta^-}{(\eta_\alpha - \eta_\beta)^2}, \quad \Xi_\alpha \left|\psi_0^{S=1}\right\rangle = 0 \quad \forall\, \alpha, \tag{1.10}$$

which we do not consider further. A Hermitian and translationally invariant annihilation operator for the $S = 1$ spin liquid state (1.5) is given by

$$H_0 = \frac{1}{2}\sum_{\alpha=1}^{N} \Omega_{\alpha}^{S=1\dagger}\Omega_{\alpha}^{S=1}. \tag{1.11}$$

Since the state is a spin singlet, i.e., invariant under SU(2) spin rotations, all the different tensor components of (1.11) must annihilate it individually. In Sect. 4.5, we obtain the desired parent Hamiltonian for the $S = 1$ spin liquid state (1.7),

$$H^{S=1} = \frac{2\pi^2}{N^2}\left[\sum_{\alpha\neq\beta}^{N} \frac{\boldsymbol{S}_\alpha\boldsymbol{S}_\beta}{|\eta_\alpha-\eta_\beta|^2} - \frac{1}{20}\sum_{\substack{\alpha,\beta,\gamma\\ \alpha\neq\beta,\gamma}}^{N} \frac{(\boldsymbol{S}_\alpha\boldsymbol{S}_\beta)(\boldsymbol{S}_\alpha\boldsymbol{S}_\gamma)+(\boldsymbol{S}_\alpha\boldsymbol{S}_\gamma)(\boldsymbol{S}_\alpha\boldsymbol{S}_\beta)}{(\bar\eta_\alpha-\bar\eta_\beta)(\eta_\alpha-\eta_\gamma)}\right], \tag{1.12}$$

by projecting out the component of H_0 which is invariant under parity, time reversal, and SU(2) spin rotations. The energy of the ground state (1.7) is given by

$$E_0^{S=1} = -\frac{2\pi^2}{N^2}\frac{N(N^2+5)}{15}. \tag{1.13}$$

Finally, we use the same methods to obtain vector annihilation operators for the $S = 1$ spin liquid state in Sect. 4.6.

In Chap. 5, we generalize the model to arbitrary spin S. We do, however, no longer start with a quantum Hall state and its parent Hamiltonian, but generalize the spin liquid states and the defining conditions for $S = \frac{1}{2}$ and $S = 1$, i.e., the conditions (1.4) and (1.9), directly to higher spins. To generalize the state vector, we first recall from Sect. 2.4.4 that the $S = 1$ spin liquid can be obtained by taking two (identical) Gutzwiller or Haldane–Shastry ground states and projecting onto the triplet or $S = 1$ configuration at each site [53]. This projection can be accomplished conveniently if we write the Haldane–Shastry ground state (2.2.3) in terms of Schwinger bosons,

$$\begin{aligned} |\psi_0^{\mathrm{HS}}\rangle &= \sum_{\{z_1,\ldots,z_M;w_1,\ldots,w_M\}} \psi_0^{\mathrm{HS}}(z_1,\ldots,z_M)\, a_{z_1}^{\dagger}\ldots a_{z_M}^{\dagger} b_{w_1}^{\dagger}\ldots b_{w_M}^{\dagger}\,|0\rangle \\ &\equiv \Psi_0^{\mathrm{HS}}[a^\dagger, b^\dagger]\,|0\rangle, \end{aligned} \tag{1.14}$$

where $M = \frac{N}{2}$ and the w_k's are those coordinates on the unit circle which are not occupied by any of the z_i's. The $S = 1$ spin liquid state (1.7) can then be written

$$|\psi_0^{S=1}\rangle = \left(\Psi_0^{\mathrm{HS}}[a^\dagger, b^\dagger]\right)^2 |0\rangle. \tag{1.15}$$

To generalize the ground state to arbitrary spin S, we just take $2S$ (identical) copies Haldane–Shastry ground state, and project at each site onto the completely symmetric representation with total spin S. In terms of Schwinger bosons,

$$|\psi_0^S\rangle = \left(\Psi_0^{\text{HS}}[a^\dagger, b^\dagger]\right)^{2S} |0\rangle. \tag{1.16}$$

This state is related to bosonic Read–Rezayi states [50] in the quantum Hall system. In Sect. 5.2, we verify that the state is annihilated by the operator

$$\Omega_\alpha^S = \sum_{\substack{\beta=1 \\ \beta\neq\alpha}}^{N} \frac{1}{\eta_\alpha - \eta_\beta} (S_\alpha^-)^{2S} S_\beta^-, \qquad \Omega_\alpha^S |\psi_0^S\rangle = 0 \ \ \forall\alpha. \tag{1.17}$$

In Sect. 5.3, we follow the same steps as for the $S = 1$ state to construct a parent Hamiltonian for the spin S state (1.16), and obtain

$$H^S = \frac{2\pi^2}{N^2} \left[\sum_{\alpha\neq\beta}^{N} \frac{\boldsymbol{S}_\alpha \boldsymbol{S}_\beta}{|\eta_\alpha - \eta_\beta|^2} - \frac{1}{2(S+1)(2S+3)} \sum_{\substack{\alpha,\beta,\gamma \\ \alpha\neq\beta,\gamma}}^{N} \frac{(\boldsymbol{S}_\alpha \boldsymbol{S}_\beta)(\boldsymbol{S}_\alpha \boldsymbol{S}_\gamma) + (\boldsymbol{S}_\alpha \boldsymbol{S}_\gamma)(\boldsymbol{S}_\alpha \boldsymbol{S}_\beta)}{(\bar{\eta}_\alpha - \bar{\eta}_\beta)(\eta_\alpha - \eta_\gamma)} \right]. \tag{1.18}$$

The energy eigenvalue is given by

$$E_0^S = -\frac{2\pi^2}{N^2} \frac{S(S+1)^2}{2S+3} \frac{N(N^2+5)}{12}. \tag{1.19}$$

This is the main result we present. In Sect. 5.4, we construct the vector annihilation operators

$$\begin{aligned} \boldsymbol{D}_\alpha^S &= \frac{1}{2} \sum_{\substack{\beta \\ \beta\neq\alpha}} \frac{\eta_\alpha + \eta_\beta}{\eta_\alpha - \eta_\beta} \left[\mathrm{i}(\boldsymbol{S}_\alpha \times \boldsymbol{S}_\beta) + (S+1)\boldsymbol{S}_\beta - \frac{1}{S+1} \boldsymbol{S}_\alpha (\boldsymbol{S}_\alpha \boldsymbol{S}_\beta) \right], \\ \boldsymbol{D}_\alpha^S |\psi_0^S\rangle &= 0 \ \ \forall\,\alpha, \end{aligned} \tag{1.20}$$

and

$$\begin{aligned} \boldsymbol{A}_\alpha^S = &\sum_{\substack{\beta \\ \beta\neq\alpha}} \frac{\boldsymbol{S}_\alpha (\boldsymbol{S}_\alpha \boldsymbol{S}_\beta) + (\boldsymbol{S}_\alpha \boldsymbol{S}_\beta) \boldsymbol{S}_\alpha + 2(S+1) \boldsymbol{S}_\beta}{|\eta_\alpha - \eta_\beta|^2} \\ &+ \sum_{\substack{\beta,\gamma \\ \beta,\gamma\neq\alpha}} \frac{1}{(\bar{\eta}_\alpha - \bar{\eta}_\beta)(\eta_\alpha - \eta_\gamma)} \\ &\cdot \left[-\frac{(\boldsymbol{S}_\alpha \boldsymbol{S}_\beta) \boldsymbol{S}_\alpha (\boldsymbol{S}_\alpha \boldsymbol{S}_\gamma) + (\boldsymbol{S}_\alpha \boldsymbol{S}_\gamma) \boldsymbol{S}_\alpha (\boldsymbol{S}_\alpha \boldsymbol{S}_\beta)}{S+1} \right. \\ &\left. + 2(S+2) \boldsymbol{S}_\alpha (\boldsymbol{S}_\beta \boldsymbol{S}_\gamma) - \boldsymbol{S}_\beta (\boldsymbol{S}_\alpha \boldsymbol{S}_\gamma) - (\boldsymbol{S}_\alpha \boldsymbol{S}_\beta) \boldsymbol{S}_\gamma \right], \\ \boldsymbol{A}_\alpha^S |\psi_0^S\rangle = &\ 0 \ \ \forall\,\alpha. \end{aligned} \tag{1.21}$$