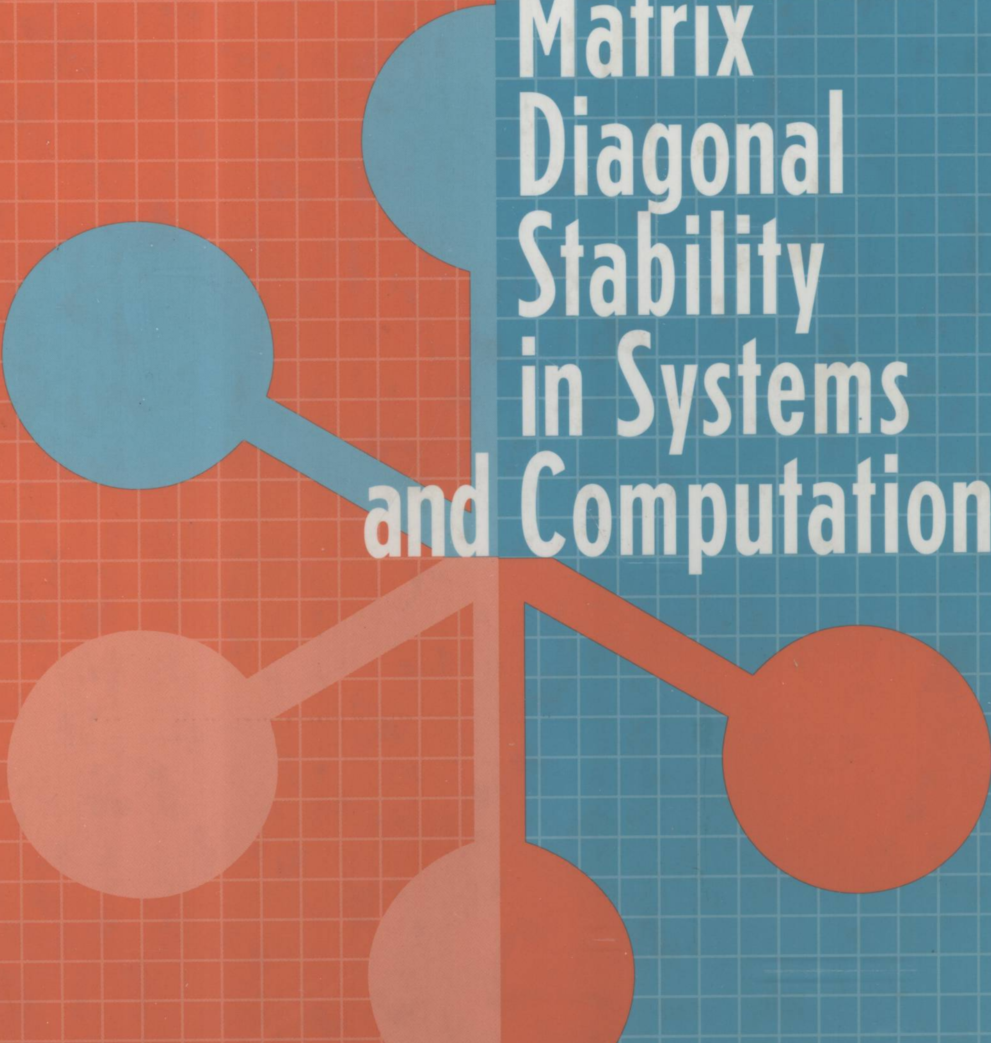
Eugenius Kaszkurewicz
Amit Bhaya

Matrix Diagonal Stability in Systems and Computation



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Eugenius Kaszkurewicz
Amit Bhaya
Department of Electrical Engineering
Federal University of Rio de Janeiro
COPPE/UFRJ
Rio de Janeiro 21945-970, Brazil

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Matrix Diagonal Stability in Systems and Computation

This book is dedicated to our families.

Preface

This monograph presents a collection of results, observations, and examples related to dynamical systems described by linear and nonlinear ordinary differential and difference equations. In particular, dynamical systems that are susceptible to analysis by the Liapunov approach are considered. The naive observation that certain "diagonal-type" Liapunov functions are ubiquitous in the literature attracted the attention of the authors and led to some natural questions. Why does this happen so often? What are the special virtues of these functions in this context? Do they occur so frequently merely because they belong to the simplest class of Liapunov functions and are thus more convenient, or are there any more specific reasons?

This monograph constitutes the authors' synthesis of the work on this subject that has been jointly developed by them, among others, producing and compiling results, properties, and examples for many years, aiming to answer these questions and also to formalize some of the folklore or "culture" that has grown around diagonal stability and diagonal-type Liapunov functions.

A natural answer to these questions would be that the use of diagonal-type Liapunov functions is frequent because of their simplicity within the class of all possible Liapunov functions. This monograph shows that, although this obvious interpretation is often adequate, there are many instances in which the Liapunov approach is best taken advantage of using diagonal-type Liapunov functions. In fact, they yield necessary and sufficient stability conditions for some classes of nonlinear dynamical systems. In other words, in many cases a diagonal-type function represents "the farthest one can go" with the Liapunov approach and has the added virtue of simplicity.

Strongly related to these diagonal-type Liapunov functions are several classes of matrices that, in most cases, describe the interconnection structure of associated dynamical systems. Chapter 2 is devoted to a discussion and presentation of the basic results for these classes of matrices; the most important being the so-called class of diagonally stable matrices. It is shown at different places in the book that in many cases the necessary and sufficient conditions for stability of nonlinear systems are, surprisingly, necessary and sufficient conditions for diagonal stability of a certain matrix associated to the nonlinear system.

There is also a strong correlation between these classes of matrices and the robustness of the dynamical systems that they are associated with. The results in Chapter 3 and the examples presented, in Chapters 4, 5, and 6, that range from neural networks and computation to passive circuits and mathematical ecology confirm this correlation.

Thus, loosely speaking, the unifying theme of this monograph is the presence and role of "diagonal stability" and the associated diagonal-type Liapunov functions in various stability aspects of certain, fairly widespread, classes of dynamical systems. Of course, there is no claim to completeness, either in terms of theoretical results on matrix diagonal and D-stability or in terms of applications; however, there is an extensive bibliography of over three hundred items. Reviewers and colleagues have often pointed out that the terms diagonal and D-stability are somewhat imprecise. However, a search made in any one of the science and technology databases, such as INSPEC or ISI's Web of Science, reveals hundreds of papers that use these terms. Thus the authors feel that it is justifiable to bow to tradition and continue to use these "traditional" terms in preference to the alternatives proposed in the literature (e.g., Volterra–Liapunov stability, Arrow–McManus stability, D^+L-**stab**, $\mathcal{D}^+$-**stab**, etc.).

Many applications of diagonal and D-stability are commented on in the sections entitled Notes and References that end every chapter and can be regarded as pointers to most of what is not touched upon in the book. These Notes also indicate and discuss the sources consulted; some others that are closely related, but have not been embedded in a deeper discussion in the text, are also included. It should also be pointed out that some long or uninformative proofs have been omitted in the interests of readability. Once again, the notes at the end of each chapter indicate where these may be found.

It is the authors' belief that the target public at which this monograph is aimed consists of graduate students and researchers in the fields of control, stability of dynamical systems, and convergence of algorithms, and that they would benefit from the contents of this book. Readers with interdisciplinary interests will also benefit from the wide range of topics from different disciplines that are included in the examples treated. The results presented are not all new, although almost all have been derived within the last two decades. The novelty of this book resides mainly in the unifying perspective provided by the matrix stability formulation of the results—in particular, matrix diagonal stability and its variants. Evidence that there is still something to be gained from this vantage point is given by the fact that interesting new results continue to crop up, as has been the case recently, for example, with neural networks and variable structure systems.

Familiarity with linear algebra and matrix theory, as well as difference and differential equations, is the mathematical background expected from the reader of this book. A prior knowledge of systems and control theory, including Liapunov stability theory, is also expected, at least in sufficient measure to provide motivation for the problems studied. Fortunately, there are many excellent books that cover this background—more details are given in the Notes and References to Chapter 1. For the reader who has a mathematical background but lacks a control background, the authors recommend a quick look at the introductory Section 1.1, which gives an overview of the book, with an attempt to explain control-related jargon to the uninitiated reader, as well as the appendices to Chapters 2 and 3, which review Liapunov and Stein equations and stability theory, respectively.

A word on notation is in order here: the Halmos symbol ■ indicates, as usual, the end of a proof of some mathematical assertion (lemma, proposition, theorem, etc.), whereas an "empty Halmos" □ indicates the end of the statement of some mathematical assertion that will not be proved explicitly in the book, either because its proof is implicit in the preceding discussion or because it will take the reader too far afield. In the latter case, "chapter and verse" citation of a reference where the proof may be found is given. Numbering of items as well as other notation is standard enough not to merit special mention here.

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Rio de Janeiro, Brazil
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Eugenius Kaszkurewicz
Amit Bhaya

Contents

1

Diagonally Stable Structures in Systems and Computation

This introductory chapter is devoted to examples that originate from different applications and that illustrate the way in which some special classes of dynamical systems dovetail with the concepts of matrix diagonal stability and the associated diagonal-type Liapunov functions.

1.1 Introduction

The class of diagonal matrices has many pleasant properties and, conversely, if a class of matrices is required to have many such properties, then, roughly speaking, it must be the diagonal class.

The objective of this book is to present classes of nonlinear dynamical systems that possess a so-called "diagonally stable structure" that is privileged in an analogous manner. Furthermore, it is also shown, by means of various examples, that this class of dynamical systems occurs in many applications in circuits and systems, in computation using asynchronous iterative methods, in control, and so on.

Terminology used throughout the book is now introduced. The term *dynamical system* refers to a set of difference or differential equations that determine the evolution of a vector in $\mathbb{R}^n$ that is also referred to as the *state vector* or simply the *state*. The evolution occurs in the *state-space* $\mathbb{R}^n$ in *discrete-time* in the case of a difference equation and in *continuous-time* in the case of a differential equation, and the path that describes this evolution in the state-space is referred to as a *(state) trajectory* or *solution.*

The term *linear dynamical system* refers to

$$x_i^+ = \sum_{j=1}^{n} a_{ij} x_j, \quad i = 1, \ldots, n \tag{1.1}$$

where the *state vector* $x = (x_1, \ldots, x_n)^T \in \mathbb{R}^n$, the *system matrix* $A = (a_{ij}) \in \mathbb{R}^{n \times n}$, and the scalar x_i^+ denotes dx_i/dt in the continuous-time case, or $x_i(t+1)$ in the discrete-time case.

A linear dynamical system with an exogenous variable, referred to as a *control* or *input vector*, is written as follows.

$$x_i^+ = \sum_{j=1}^{n} a_{ij} x_j + \sum_{k=1}^{m} b_{ik} u_k, \quad i = 1, \ldots, n \tag{1.2}$$

The adjective control used for the vector u arises from the fact that it can be chosen so as to control the evolution in time of the state vector $x(t)$, i.e., the trajectory of the system. In fact, in *linear state feedback control,* the input vector u is chosen as a linear function of the state, $u = Kx$, so that the controlled system becomes $x^+ = (A + BK)x$, which is again in the form of system (1.1), although the system matrix has changed from A to $A + BK$. The matrix $A + BK$ is referred to as the *closed-loop* system matrix. Such a choice of a *feedback matrix* K is known as *state feedback stabilization* when stability of the system is the desired property.

To explain the term *diagonally stable,* consider the matrix equations:

$$A^T P + PA = -Q \tag{1.3}$$

$$A^T PA - P = -Q \tag{1.4}$$

The equation (1.3) (respectively, (1.4)) is referred to as the Liapunov (respectively Stein) equation in A. If there exist positive definite matrices P and Q satisfying the equation (1.3) (respectively (1.4)), then the matrix A is said to be *continuous-time* or *Hurwitz stable* (respectively *discrete-time* or *Schur stable*). Another well known characterization of stability is in terms of the eigenvalues of the matrix A: If all eigenvalues of A lie in the open left half complex plane (i.e., all have negative real parts), then A is Hurwitz stable; if all eigenvalues lie within the open unit disk in the complex plane, then A is Schur stable. The quadratic form $x^T Px$ is referred to as the *quadratic Liapunov function* associated to system (1.1) and traditionally denoted $V(x)$. The stability being referred to is asymptotic stability: Namely, if the initial condition is nonzero, the resulting trajectory of (1.1) goes to the zero solution asymptotically. If the positive definite solution P is, in addition, *diagonal,* then, in the continuous-time case, the matrix A is referred to as *Hurwitz diagonally stable* and the quadratic form $V(x) = x^T Px$ as the associated diagonal quadratic Liapunov function.

Such diagonally stable matrices and the associated diagonal quadratic Liapunov functions crop up in the stability analysis of a wide variety of systems: variable structure systems, digital filters, Lotka–Volterra systems in mathematical ecology—to name a few examples. This is because, in these examples, the underlying mathematical models can all be considered to be special cases of the following set of n nonlinear differential or difference equations:

$$x_i^+ = \sum_{j=1}^{n} a_{ij}\Phi_{ij}(x,t), \quad i = 1,\ldots,n \tag{1.5}$$

where for all i and j, $\Phi_{ij} : \mathbb{R}^n \times \mathbb{T} \to \mathbb{R}^n$ satisfy certain so-called sector conditions, and, as before, x_i^+ denotes dx_i/dt in the continuous-time case, when $\mathbb{T} = \mathbb{R}^+$; or $x_i(t+1)$ in the discrete-time case, when $\mathbb{T} = \mathbb{N}$.

As first observed by Persidskii [Per69], for a special case of (1.5), such models admit a class of *diagonal-type Liapunov functions.* These are functions that are, in the simplest case, quadratic Liapunov functions of the the type $x^T P x$, where x is a vector and P a diagonal matrix, with all diagonal entries positive, referred to briefly as a positive diagonal matrix. More generally, these are functions that are conducive to the use of matrix diagonal stability in proving that they decrease along the trajectories of the dynamical system in question. A historical note appropriate here is that the first use of a diagonal-type Liapunov function seems to have been made by Volterra in his classic studies of fish populations in the Adriatic [Vol31].

In the discrete-time case, for the class of diagonal quadratic Liapunov functions, a certain converse result holds for a class of linear time-varying systems whose trajectories are the same as those of (1.5); namely, that if the whole class of linear time-varying systems is to be stable, then the matrix $|A| := (|a_{ij}|)$ must admit a diagonal solution to its Stein equation, i.e., must be diagonally stable. Equivalently, the class of systems in question admits a simultaneous diagonal quadratic Liapunov function.

In several applications, a variation of the concept of diagonal stability also arises—this is the concept of additive diagonal stability. A matrix A is called *additively diagonally stable* if $A + D$ is diagonally stable for any nonpositive diagonal matrix D.

Informally, a dynamical system that can be written, perhaps after a change of coordinates, in the form (1.5) with the matrix A diagonally stable, is said to possess a *diagonally stable structure.* There are also several variants of equation (1.5), presented in Chapter 3, that are also susceptible to analysis by diagonal-type Liapunov functions, using the concept of additive diagonal stability, thereby broadening considerably the class of diagonally stable structures.

Two closely related terms that are frequently used in the book to qualify stability are *robust* and *absolute.* Both terms apply to a specified class of dynamical systems, denoted $\mathcal{C}$. Stability of a given dynamical system is said to be *robust* if all its neighbors in $\mathcal{C}$, defined in some suitable sense, are stable. The dynamical system is then said to possess the property of *robustness*. A criterion is called an *absolute stability condition* if it ensures that *all* members of the class $\mathcal{C}$ are stable. In this sense, absolute stability is an extreme form of robust stability in which it is required that all members of a class be stable, rather than just some neighborhood. Suppose, for example, that a dynamical system S is defined for each value of a parameter p and that $\mathcal{C} := \{S(p) : p \in \mathcal{P}\}$, for some open set of parameters $\mathcal{P}$. Then a given dynamical system $S(p_0) \in \mathcal{C}$ is said to be *robustly stable* if there is some neighborhood of p_0, N_{p_0}, contained in $\mathcal{P}$, such that $S(p)$ is stable for all $p \in N_{p_0}$. A stability condition is said to be robust if it guarantees the stability of a given dynamical system, usually called the *nominal system*, as well as that of some set of its neighbors. An absolute stability condition guarantees the stability of a whole class of systems. Of course, if the whole class turns out to be exactly the set of neighbors or vice versa, then the two definitions coincide. An interesting historical observation is that the term absolute stability originated in the Russian literature and still tends to be preferred in this literature. The term robust originated around 1972 in the Western control literature, borrowed from the statistics literature (see the preface to [Dor87]), in which it tends to be preferred to the term absolute. Prior to 1972, the literature used terms such as stability of *uncertain* systems, *sensitivity* or *roughness* of stability, etc. An important extension of the discrete-time version of (1.5) includes time-varying delays on the right-hand side and can be written as follows.

$$x_i(k+1) = \sum_{j=1}^{n} A_{ij}(x_1(k), \ldots, x_n(k))x_j(d_{ij}(k)), \tag{1.6}$$

for $i = 1, 2, \cdots, n$, $k = 0, 1, 2, \cdots$, and where $d_{ij}(k) \in \{k, k-1, k-2, \cdots, k-d\}$, for all k, for some integer $d > 0$ and for $i, j = 1, \cdots, n$.

Diagonal-type Liapunov functions also allow simple proofs of the asymptotic stability of neural circuits and asynchronous iterative computations and systems with delays since the dynamical equations describing these systems can be put in the form (1.6), with appropriate definitions of the matrices A_{ij}. However, for these systems, *quadratic* diagonal Liapunov functions do not always suffice to show stability and more general diagonal-type functions must usually be resorted to.

Diagonally stable structures arise in another context where the question of robustness of stability can be reduced to that of the invariance of asymptotic stability of a certain matrix under multiplication by certain subclasses of the class of positive diagonal matrices. In the continuous-time case, if the stability of a given matrix A is maintained under premultiplication by a diagonal matrix with positive diagonal entries, then A is said to be *Hurwitz D-stable* and this type of matrix stability is called Hurwitz D-stability. It was first studied in a price stability problem in the continuous-time case in economics. Since then applications have surfaced in a variety of fields, but the problem of effective characterization of D-stability has so far resisted solution. More details on this as well as on the connections between diagonal and D-stability can be found in Chapter 2.

The other sections of this chapter present a series of examples that serve to illustrate these introductory remarks, and also to motivate the reader to get interested in the book. Some remarks on the examples in this chapter and others elsewhere in the book are worth making. Several of these examples have been much studied in the literature and the results shown here are not necessarily new, but are put in a perspective where certain structural features are taken advantage of in order to get an understanding of the solution to the problem posed in each example. To this end, simple examples have been chosen. However, if readers encounter some difficulty in understanding a particular example completely, they should observe that an effort has been made to provide cross references to the chapters in which a more complete explanation of the ideas involved in the examples is given.

1.2 Robust Stability of a Mechanical System

Consider a simple mechanical device that consists of a disc fixed to a massless elastic shaft, with stiffness coefficient k, at the center. Assume that friction is internal to the shaft. Let the damping coefficient be denoted f, and let y_1 and y_2 represent the deflections of the disk, regarded as a particle of mass m located at the center of the shaft. Then, in a coordinate system rotating at the angular velocity of the shaft, the linearized equations of motion of the device are as follows [Zie68].

$$\begin{aligned} m\ddot{y}_1 + f\dot{y}_1 - 2m\omega\dot{y}_2 + (k - m\omega)y_1 &= 0, \\ m\ddot{y}_2 + f\dot{y}_2 + 2m\omega\dot{y}_1 + (k - m\omega)y_2 &= 0. \end{aligned} \tag{1.7}$$

Assume that the parameters are normalized so that $k/m = f/m = 1$ and let $x := (y_1, \dot{y}_1, y_2, \dot{y}_2)^T \in \mathbb{R}^4$. The dynamics of this mechanical system is written in the state-space form as follows.

$$\frac{dx}{dt} = A(\omega)x \tag{1.8}$$

where $x \in \mathbb{R}^4$, the angular velocity ω is considered to be a perturbation parameter with respect to the nominal value $\omega = 0$, and the matrix $A(\omega)$ has the form below:

$$A(\omega) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ \omega^2 - 1 & -1 & 0 & 2\omega \\ 0 & 0 & 0 & 1 \\ 0 & -2\omega & \omega^2 - 1 & -1 \end{bmatrix} \tag{1.9}$$