Basic Algebra II

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Preface

This volume is a text for a second course in algebra that presupposes an introductory course covering the type of material contained in the Introduction and the first three or four chapters of *Basic Algebra* I. These chapters dealt with the rudiments of set theory, group theory, rings, modules, especially modules over a principal ideal domain, and Galois theory focused on the classical problems of solvability of equations by radicals and constructions with straight-edge and compass.

Basic Algebra II contains a good deal more material than can be covered in a year's course. Selection of chapters as well as setting limits within chapters will be essential in designing a realistic program for a year. We briefly indicate several alternatives for such a program: Chapter 1 with the addition of section 2.9 as a supplement to section 1.5, Chapters 3 and 4, Chapter 6 to section 6.11, Chapter 7 to section 7.13, sections 8.1–8.3, 8.6, 8.12, Chapter 9 to section 9.13. A slight modification of this program would be to trade off sections 4.6–4.8 for sections 5.1–5.5 and 5.9. For students who have had no Galois theory it will be desirable to supplement section 8.3 with some of the material of Chapter 4 of *Basic Algebra* I. If an important objective of a course in algebra is an understanding of the foundations of algebraic structures and the

relation between algebra and mathematical logic, then all of Chapter 2 should be included in the course. This, of course, will necessitate thinning down other parts, e.g., homological algebra. There are many other possibilities for a one-year course based on this text.

The material in each chapter is treated to a depth that permits the use of the text also for specialized courses. For example, Chapters 3, 4, and 5 could constitute a one-semester course on representation theory of finite groups, and Chapter 7 and parts of Chapters 8, 9, and 10 could be used for a one-semester course in commutative algebras. Chapters 1, 3, and 6 could be used for an introductory course in homological algebra.

Chapter 11 on real fields is somewhat isolated from the remainder of the book. However, it constitutes a direct extension of Chapter 5 of *Basic Algebra* I and includes a solution of Hilbert's problem on positive semi-definite rational functions, based on a theorem of Tarski's that was proved in Chapter 5 of the first volume. Chapter 11 also includes Pfister's beautiful theory of quadratic forms that gives an answer to the question of the minimum number of squares required to express a sum of squares of rational functions of n real variables (see section 11.5).

Aside from its use as a text for a course, the book is designed for independent reading by students possessing the background indicated. A great deal of material is included. However, we believe that nearly all of this is of interest to mathematicians of diverse orientations and not just to specialists in algebra. We have kept in mind a general audience also in seeking to reduce to a minimum the technical terminology and in avoiding the creation of an overly elaborate machinery before presenting the interesting results. Occasionally we have had to pay a price for this in proofs that may appear a bit heavy to the specialist.

Many exercises have been included in the text. Some of these state interesting additional results, accompanied with sketches of proofs. Relegation of these to the exercises was motivated simply by the desire to reduce the size of the text somewhat. The reader would be well advised to work a substantial number of the exercises.

An extensive bibliography seemed inappropriate in a text of this type. In its place we have listed at the end of each chapter one or two specialized texts in which the reader can find extensive bibliographies on the subject of the chapter. Occasionally, we have included in our short list of references one or two papers of historical importance. None of this has been done in a systematic or comprehensive manner.

Again it is a pleasure for me to acknowledge the assistance of many friends in suggesting improvements of earlier versions of this text. I should mention first the students whose perceptions detected flaws in the exposition and sometimes suggested better proofs that they had seen elsewhere. Some of the students

who have contributed in this way are Monica Barattieri, Ying Cheng, Daniel Corro, William Ellis, Craig Huneke, and Kenneth McKenna. Valuable suggestions have been communicated to me by Professors Kevin McCrimmon, James D. Reid, Robert L. Wilson, and Daniel Zelinsky. I have received such suggestions also from my colleagues Professors Walter Feit, George Seligman, and Tsuneo Tamagawa. The arduous task of proofreading was largely taken over by Ying Cheng, Professor Florence Jacobson, and James Reid. Florence Jacobson assisted in compiling the index. Finally we should mention the fine job of typing that was done by Joyce Harry and Donna Belli. I am greatly indebted to all of these individuals, and I take this opportunity to offer them my sincere thanks.

January 1980 *Nathan Jacobson*

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Contents of Basic Algebra I

Introduction

In the Introduction to *Basic Algebra* I (abbreviated throughout as "BAI") we gave an account of the set theoretic concepts that were needed for that volume. These included the power set $\mathcal{P}(S)$ of a set S, the Cartesian product $S_1 \times S_2$ of two sets S_1 and S_2, maps (=functions), and equivalence relations. In the first volume we generally gave preference to constructive arguments and avoided transfinite methods altogether.

The results that are presented in this volume require more powerful tools, particularly for the proofs of certain existence theorems. Many of these proofs will be based on a result, called Zorn's lemma, whose usefulness for proving such existence theorems was first noted by Max Zorn. We shall require also some results on the arithmetic of cardinal numbers. All of this fits into the framework of the Zermelo–Fraenkel axiomatization of set theory, including the axiom of choice (the so-called ZFC set theory). Two excellent texts that can be used to fill in the details omitted in our discussion are P. R. Halmos' *Naive Set Theory* and the more substantial *Set Theory and the Continuum Hypothesis* by P. J. Cohen.

Classical mathematics deals almost exclusively with structures based on sets. On the other hand, category theory—which will be introduced in Chapter 1—

deals with collections of sets, such as all groups, that need to be treated differently from sets. Such collections are called classes. A brief indication of a suitable foundation for category theory is given in the last section of this Introduction.

0.1 ZORN'S LEMMA

We shall now formulate a maximum principle of set theory called Zorn's lemma. We state this first for subsets of a given set. We recall that a set C of subsets of a set S (that is, a subset of the power set $\mathscr{P}(S)$) is called a *chain* if C is totally ordered by inclusion, that is, for any $A, B \in C$ either $A \subset B$ or $B \subset A$. A set T of subsets of S is called *inductive* if the union $\bigcup A_\alpha$ of any chain $C = \{A_\alpha\} \subset T$ is a member of T. We can now state

ZORN'S LEMMA (First formulation). *Let T be a non-vacuous set of subsets of a set S. Assume T is inductive. Then T contains a maximal element, that is, there exists an $M \in T$ such that no $A \in T$ properly contains M.*

There is another formulation of Zorn's lemma in terms of partially ordered sets (BAI, p. 434). Let $P, \geqslant$ be a partially ordered set. We call $P, \geqslant$ (*totally* or *linearly*) *ordered* if for every $a, b \in P$ either $a \geqslant b$ or $b \geqslant a$. We call P *inductive* if every non-vacuous subset C of P that is (totally) ordered by $\geqslant$ as defined in P has a least upper bound in P, that is, there exists a $u \in P$ such that $u \geqslant a$ for every $a \in C$ and if $v \geqslant a$ for every $a \in C$ then $v \geqslant u$. Then we have

ZORN'S LEMMA (Second formulation). *Let $P, \geqslant$ be a partially ordered set that is inductive. Then P contains maximal elements, that is, there exists $m \in P$ such that no $a \in P$ satisfies $m < a$.*

It is easily seen that the two formulations of Zorn's lemma are equivalent, so there is no harm in referring to either as "Zorn's lemma." It can be shown that Zorn's lemma is equivalent to the

AXIOM OF CHOICE. *Let S be a set, $\mathscr{P}(S)^*$ the set of non-vacuous subsets of S. Then there exists a map f (a "choice function") of $\mathscr{P}(S)^*$ into S such that $f(A) \in A$ for every $A \in \mathscr{P}(S)^*$.*

This is equivalent also to the following: If $\{A_\alpha\}$ is a set of non-vacuous sets A_α, then the Cartesian product $\prod A_\alpha \neq \varnothing$.

The statement that the axiom of choice implies Zorn's lemma can be proved