



DOUGLAS R. HOFSTADTER

GÖDEL, ESCHER, BACH:

AN ETERNAL GOLDEN BRAID

A METAPHORICAL FUGUE ON MINDS AND MACHINES
IN THE SPIRIT OF LEWIS CARROLL



GÖDEL,

ESCHER,

BACH:

an Eternal Golden Braid

Douglas R. Hofstadter



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Overview

Part I: GEB

Introduction: A Musico-Logical Offering. The book opens with the story of Bach's *Musical Offering*. Bach made an impromptu visit to King Frederick the Great of Prussia, and was requested to improvise upon a theme presented by the King. His improvisations formed the basis of that great work. The *Musical Offering* and its story form a theme upon which I "improvise" throughout the book, thus making a sort of "Metamusical Offering". Self-reference and the interplay between different levels in Bach are discussed; this leads to a discussion of parallel ideas in Escher's drawings and then Gödel's Theorem. A brief presentation of the history of logic and paradoxes is given as background for Gödel's Theorem. This leads to mechanical reasoning and computers, and the debate about whether Artificial Intelligence is possible. I close with an explanation of the origins of the book—particularly the why and wherefore of the Dialogues.

Three-Part Invention. Bach wrote fifteen three-part inventions. In this three-part Dialogue, the Tortoise and Achilles—the main fictional protagonists in the Dialogues—are "invented" by Zeno (as in fact they were, to illustrate Zeno's paradoxes of motion). Very short, it simply gives the flavor of the Dialogues to come.

Chapter I: The MU-puzzle. A simple formal system (the MIU-system) is presented, and the reader is urged to work out a puzzle to gain familiarity with formal systems in general. A number of fundamental notions are introduced: string, theorem, axiom, rule of inference, derivation, formal system, decision procedure, working inside/outside the system.

Two-Part Invention. Bach also wrote fifteen two-part inventions. This two-part Dialogue was written not by me, but by Lewis Carroll in 1895. Carroll borrowed Achilles and the Tortoise from Zeno, and I in turn borrowed them from Carroll. The topic is the relation between reasoning, reasoning about reasoning, reasoning about reasoning about reasoning, and so on. It parallels, in a way, Zeno's paradoxes about the impossibility of motion, seeming to show, by using infinite regress, that reasoning is impossible. It is a beautiful paradox, and is referred to several times later in the book.

Chapter II: Meaning and Form in Mathematics. A new formal system (the pq-system) is presented, even simpler than the MIU-system of Chapter I. Apparently meaningless at first, its symbols are suddenly revealed to possess meaning by virtue of the form of the theorems they appear in. This revelation is the first important insight into meaning: its deep connection to isomorphism. Various issues related to meaning are then discussed, such as truth, proof, symbol manipulation, and the elusive concept, "form".

Sonata for Unaccompanied Achilles. A Dialogue which imitates the Bach Sonatas for unaccompanied violin. In particular, Achilles is the only speaker, since it is a transcript of one end of a telephone call, at the far end of which is the Tortoise. Their conversation concerns the concepts of "figure" and "ground" in various

contexts—e.g., Escher’s art. The Dialogue itself forms an example of the distinction, since Achilles’ lines form a “figure”, and the Tortoise’s lines—implicit in Achilles’ lines—form a “ground”.

Chapter III: Figure and Ground. The distinction between figure and ground in art is compared to the distinction between theorems and nontheorems in formal systems. The question “Does a figure necessarily contain the same information as its ground?” leads to the distinction between recursively enumerable sets and recursive sets.

Contracrostipunctus. This Dialogue is central to the book, for it contains a set of paraphrases of Gödel’s self-referential construction and of his Incompleteness Theorem. One of the the paraphrases of the Theorem says, “For each record player there is a record which it cannot play.” The Dialogue’s title is a cross between the word “acrostic” and the word “contrapunctus”, a Latin word which Bach used to denote the many fugues and canons making up his *Art of the Fugue*. Some explicit references to the *Art of the Fugue* are made. The Dialogue itself conceals some acrostic tricks.

Chapter IV: Consistency, Completeness, and Geometry. The preceding Dialogue is explicated to the extent it is possible at this stage. This leads back to the question of how and when symbols in a formal system acquire meaning. The history of Euclidean and non-Euclidean geometry is given, as an illustration of the elusive notion of “undefined terms”. This leads to ideas about the consistency of different and possibly “rival” geometries. Through this discussion the notion of undefined terms is clarified, and the relation of undefined terms to perception and thought processes is considered.

Little Harmonic Labyrinth. This is based on the Bach organ piece by the same name. It is a playful introduction to the notion of recursive—i.e., nested—structures. It contains stories within stories. The frame story, instead of finishing as expected, is left open, so the reader is left dangling without resolution. One nested story concerns modulation in music—particularly an organ piece which ends in the wrong key, leaving the listener dangling without resolution.

Chapter V: Recursive Structures and Processes. The idea of recursion is presented in many different contexts: musical patterns, linguistic patterns, geometric structures, mathematical functions, physical theories, computer programs, and others.

Canon by Intervallic Augmentation. Achilles and the Tortoise try to resolve the question, “Which contains more information—a record, or the phonograph which plays it?” This odd question arises when the Tortoise describes a single record which, when played on a set of different phonographs, produces two quite different melodies: B-A-C-H and C-A-G-E. It turns out, however, that these melodies are “the same”, in a peculiar sense.

Chapter VI: The Location of Meaning. A broad discussion of how meaning is split among coded message, decoder, and receiver. Examples presented include strands of DNA, undeciphered inscriptions on ancient tablets, and phonograph records sailing out in space. The relationship of intelligence to “absolute” meaning is postulated.

Chromatic Fantasy, And Feud. A short Dialogue bearing hardly any resemblance, except in title, to Bach’s *Chromatic Fantasy and Fugue*. It concerns the proper way to manipulate sentences so as to preserve truth—and in particular the question

of whether there exist rules for the usage of the word “and”. This Dialogue has much in common with the Dialogue by Lewis Carroll.

Chapter VII: The Propositional Calculus. It is suggested how words such as “and” can be governed by formal rules. Once again, the ideas of isomorphism and automatic acquisition of meaning by symbols in such a system are brought up. All the examples in this Chapter, incidentally, are “Zentences”—sentences taken from Zen kōans. This is purposefully done, somewhat tongue-in-cheek, since Zen kōans are deliberately illogical stories.

Crab Canon. A Dialogue based on a piece by the same name from the *Musical Offering*. Both are so named because crabs (supposedly) walk backwards. The Crab makes his first appearance in this Dialogue. It is perhaps the densest Dialogue in the book in terms of formal trickery and level-play. Gödel, Escher, and Bach are deeply intertwined in this very short Dialogue.

Chapter VIII: Typographical Number Theory. An extension of the Propositional Calculus called “TNT” is presented. In TNT, number-theoretical reasoning can be done by rigid symbol manipulation. Differences between formal reasoning and human thought are considered.

A Mu Offering. This Dialogue foreshadows several new topics in the book. Ostensibly concerned with Zen Buddhism and kōans, it is actually a thinly veiled discussion of theoremhood and nontheoremhood, truth and falsity, of strings in number theory. There are fleeting references to molecular biology—particularly the Genetic Code. There is no close affinity to the *Musical Offering*, other than in the title and the playing of self-referential games.

Chapter IX: Mumon and Gödel. An attempt is made to talk about the strange ideas of Zen Buddhism. The Zen monk Mumon, who gave well known commentaries on many kōans, is a central figure. In a way, Zen ideas bear a metaphorical resemblance to some contemporary ideas in the philosophy of mathematics. After this “Zennery”, Gödel’s fundamental idea of Gödel-numbering is introduced, and a first pass through Gödel’s Theorem is made.

Part II: EGB

Prelude . . . This Dialogue attaches to the next one. They are based on preludes and fugues from Bach’s *Well-Tempered Clavier*. Achilles and the Tortoise bring a present to the Crab, who has a guest: the Anteater. The present turns out to be a recording of the *W.T.C.*; it is immediately put on. As they listen to a prelude, they discuss the structure of preludes and fugues, which leads Achilles to ask how to hear a fugue: as a whole, or as a sum of parts? This is the debate between holism and reductionism, which is soon taken up in the *Ant Fugue*.

Chapter X: Levels of Description, and Computer Systems. Various levels of seeing pictures, chessboards, and computer systems are discussed. The last of these is then examined in detail. This involves describing machine languages, assembly languages, compiler languages, operating systems, and so forth. Then the discussion turns to composite systems of other types, such as sports teams, nuclei, atoms, the weather, and so forth. The question arises as to how many intermediate levels exist—or indeed whether any exist.

... *Ant Fugue*. An imitation of a musical fugue: each voice enters with the same statement. The theme—holism versus reductionism—is introduced in a recursive picture composed of words composed of smaller words, etc. The words which appear on the four levels of this strange picture are “HOLISM”, “REDUCTIONISM”, and “MU”. The discussion veers off to a friend of the Anteater’s—Aunt Hillary, a conscious ant colony. The various levels of her thought processes are the topic of discussion. Many fugal tricks are ensconced in the Dialogue. As a hint to the reader, references are made to parallel tricks occurring in the fugue on the record to which the foursome is listening. At the end of the *Ant Fugue*, themes from the *Prelude* return, transformed considerably.

Chapter XI: Brains and Thoughts. “How can thoughts be supported by the hardware of the brain?” is the topic of the Chapter. An overview of the large-scale and small-scale structure of the brain is first given. Then the relation between concepts and neural activity is speculatively discussed in some detail.

English French German Suite. An interlude consisting of Lewis Carroll’s nonsense poem “Jabberwocky” together with two translations: one into French and one into German, both done last century.

Chapter XII: Minds and Thoughts. The preceding poems bring up in a forceful way the question of whether languages, or indeed minds, can be “mapped” onto each other. How is communication possible between two separate physical brains? What do all human brains have in common? A geographical analogy is used to suggest an answer. The question arises, “Can a brain be understood, in some objective sense, by an outsider?”

Aria with Diverse Variations. A Dialogue whose form is based on Bach’s *Goldberg Variations*, and whose content is related to number-theoretical problems such as the Goldbach conjecture. This hybrid has as its main purpose to show how number theory’s subtlety stems from the fact that there are many diverse variations on the theme of searching through an infinite space. Some of them lead to infinite searches, some of them lead to finite searches, while some others hover in between.

Chapter XIII: BlooP and FlooP and Gloop. These are the names of three computer languages. BlooP programs can carry out only predictably finite searches, while FlooP programs can carry out unpredictable or even infinite searches. The purpose of this Chapter is to give an intuition for the notions of primitive recursive and general recursive functions in number theory, for they are essential in Gödel’s proof.

Air on G’s String. A Dialogue in which Gödel’s self-referential construction is mirrored in words. The idea is due to W. V. O. Quine. This Dialogue serves as a prototype for the next Chapter.

Chapter XIV: On Formally Undecidable Propositions of TNT and Related Systems. This Chapter’s title is an adaptation of the title of Gödel’s 1931 article, in which his Incompleteness Theorem was first published. The two major parts of Gödel’s proof are gone through carefully. It is shown how the assumption of consistency of TNT forces one to conclude that TNT (or any similar system) is incomplete. Relations to Euclidean and non-Euclidean geometry are discussed. Implications for the philosophy of mathematics are gone into with some care.

Birthday Cantatatata . . . In which Achilles cannot convince the wily and skeptical Tortoise that today is his (Achilles') birthday. His repeated but unsuccessful tries to do so foreshadow the repeatability of the Gödel argument.

Chapter XV: Jumping out of the System. The repeatability of Gödel's argument is shown, with the implication that TNT is not only incomplete, but "essentially incomplete". The fairly notorious argument by J. R. Lucas, to the effect that Gödel's Theorem demonstrates that human thought cannot in any sense be "mechanical", is analyzed and found to be wanting.

Edifying Thoughts of a Tobacco Smoker. A Dialogue treating of many topics, with the thrust being problems connected with self-replication and self-reference. Television cameras filming television screens, and viruses and other subcellular entities which assemble themselves, are among the examples used. The title comes from a poem by J. S. Bach himself, which enters in a peculiar way.

Chapter XVI: Self-Ref and Self-Rep. This Chapter is about the connection between self-reference in its various guises, and self-reproducing entities (e.g., computer programs or DNA molecules). The relations between a self-reproducing entity and the mechanisms external to it which aid it in reproducing itself (e.g., a computer or proteins) are discussed—particularly the fuzziness of the distinction. How information travels between various levels of such systems is the central topic of this Chapter.

The Magnificrab, Indeed. The title is a pun on Bach's *Magnificat* in D. The tale is about the Crab, who gives the appearance of having a magical power of distinguishing between true and false statements of number theory by reading them as musical pieces, playing them on his flute, and determining whether they are "beautiful" or not.

Chapter XVII: Church, Turing, Tarski, and Others. The fictional Crab of the preceding Dialogue is replaced by various real people with amazing mathematical abilities. The Church-Turing Thesis, which relates mental activity to computation, is presented in several versions of differing strengths. All are analyzed, particularly in terms of their implications for simulating human thought mechanically, or programming into a machine an ability to sense or create beauty. The connection between brain activity and computation brings up some other topics: the halting problem of Turing, and Tarski's Truth Theorem.

SHRDLU, Toy of Man's Designing. This Dialogue is lifted out of an article by Terry Winograd on his program SHRDLU; only a few names have been changed. In it, a program communicates with a person about the so-called "blocks world" in rather impressive English. The computer program appears to exhibit some real understanding—in its limited world. The Dialogue's title is based on *Jesu, Joy of Man's Desiring*, one movement of Bach's Cantata 147.

Chapter XVIII: Artificial Intelligence: Retrospects. This Chapter opens with a discussion of the famous "Turing test"—a proposal by the computer pioneer Alan Turing for a way to detect the presence or absence of "thought" in a machine. From there, we go on to an abridged history of Artificial Intelligence. This covers programs that can—to some degree—play games, prove theorems, solve problems, compose music, do mathematics, and use "natural language" (e.g., English).

Contrafactus. About how we unconsciously organize our thoughts so that we can imagine hypothetical variants on the real world all the time. Also about aberrant variants of this ability—such as possessed by the new character, the Sloth, an avid lover of French fries, and rabid hater of counterfactuals.

Chapter XIX: Artificial Intelligence: Prospects. The preceding Dialogue triggers a discussion of how knowledge is represented in layers of contexts. This leads to the modern AI idea of “frames”. A frame-like way of handling a set of visual pattern puzzles is presented, for the purpose of concreteness. Then the deep issue of the interaction of concepts in general is discussed, which leads into some speculations on creativity. The Chapter concludes with a set of personal “Questions and Speculations” on AI and minds in general.

Sloth Canon. A canon which imitates a Bach canon in which one voice plays the same melody as another, only upside down and twice as slowly, while a third voice is free. Here, the Sloth utters the same lines as the Tortoise does, only negated (in a liberal sense of the term) and twice as slowly, while Achilles is free.

Chapter XX: Strange Loops, Or Tangled Hierarchies. A grand windup of many of the ideas about hierarchical systems and self-reference. It is concerned with the snarls which arise when systems turn back on themselves—for example, science probing science, government investigating governmental wrongdoing, art violating the rules of art, and finally, humans thinking about their own brains and minds. Does Gödel’s Theorem have anything to say about this last “snarl”? Are free will and the sensation of consciousness connected to Gödel’s Theorem? The Chapter ends by tying Gödel, Escher, and Bach together once again.

Six-Part Ricercar. This Dialogue is an exuberant game played with many of the ideas which have permeated the book. It is a reenactment of the story of the *Musical Offering*, which began the book; it is simultaneously a “translation” into words of the most complex piece in the *Musical Offering*: the *Six-Part Ricercar*. This duality imbues the Dialogue with more levels of meaning than any other in the book. Frederick the Great is replaced by the Crab, pianos by computers, and so on. Many surprises arise. The Dialogue’s content concerns problems of mind, consciousness, free will, Artificial Intelligence, the Turing test, and so forth, which have been introduced earlier. It concludes with an implicit reference to the beginning of the book, thus making the book into one big self-referential loop, symbolizing at once Bach’s music, Escher’s drawings, and Gödel’s Theorem.

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Words of Thanks

This book was brewing in my mind over a period of probably nearly twenty years—ever since I was thirteen and thinking about how I thought in English and French. And even before that, there were clear signs of my main-line interest. I remember that at some early age, there was nothing more fascinating to me than the idea of taking three 3's: operating upon 3 with *itself*! I was sure that this idea was so subtle that it was inconceivable to anyone else—but I dared ask my mother one day how big it was anyway, and she answered “Nine”. I wasn't sure she knew what I meant, though. Later, my father initiated me into the mysteries of square roots, and *i* . . .

I owe more to my parents than to anyone else. They have been pillars of support I could rely on at all times. They have guided me, inspired me, encouraged me, and sustained me. Most of all, they have always believed in me. It is to them that this book is dedicated.

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In a way, this book is a statement of my religion. I hope that this will come through to my readers, and that my enthusiasm and reverence for certain ideas will infiltrate the hearts and minds of a few people. That is the best I could ask for.

D. R. H.
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