Second Edition

TRIGONOMETRY

A Modern Approach

ELICH · ELICH · CANNON

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TRIGONOMETRY

A Modern Approach

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ALGEBRA FORMULAS

Factoring Identities

$ax + bx = (a + b)x$

$a^2 \pm 2ab + b^2 = (a \pm b)^2$

$a^2 - b^2 = (a - b)(a + b)$

$a^3 \pm b^3 = (a \pm b)(a^2 \mp ab + b^2)$

Quadratic Formula

If $ax^2 + bx + c = 0$, where $a \neq 0$, then

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Exponents

If $b > 0$ and u, v, t are any real numbers, then

$b^u \cdot b^v = b^{u+v}$

$b^u / b^v = b^{u-v}$

$(b^u)^t = b^{ut}$

$b^0 = 1$

$b^{-u} = 1/b^u$

Logarithms

Metric

Linear measure

1 meter = 1 m = 39.37 inches
1 kilometer = 1 km = 0.62137 miles
1 mm = 0.001 m
1 cm = 0.01 m
1 km = 1000 m

Volume measure

1 liter = 1 l = 1.057 quarts
1 l = 1000 cm^3
1 ml = 0.001 l
1 kl = 1000 l

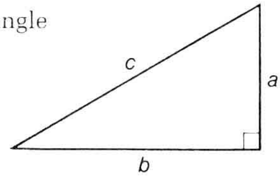
1 kg = 1000 g

GEOMETRY FORMULAS

Pythagorean Theorem

For a right triangle

$a^2 + b^2 = c^2$

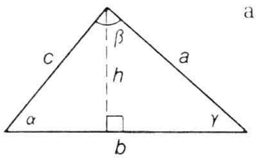


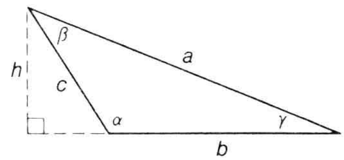
Triangle

$\alpha + \beta + \gamma = 180°$

area $= \frac{1}{2}bh$

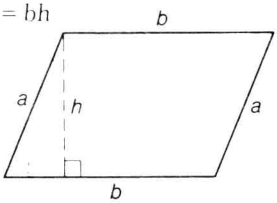
perimeter $= a + b + c$





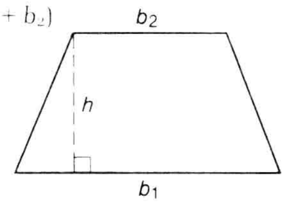
Parallelogram

area $= bh$



Trapezoid

area $= \frac{1}{2}h\,(b_1 + b_2)$



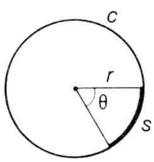
Circle

circumference $= 2\pi r$

area $= \pi r^2$

$s = r\theta$

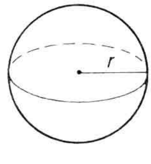
area of sector $= \frac{1}{2}r^2\theta$



Sphere

surface area $= 4\pi r^2$

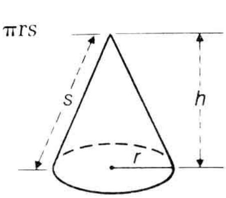
volume $= \frac{4}{3}\pi r^3$



Cone (right circular)

lateral surface $= \pi rs$

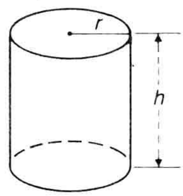
volume $= \frac{1}{3}\pi r^2 h$



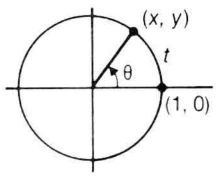
Cylinder (right circular)

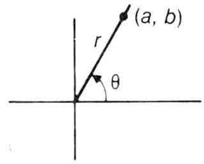
lateral surface $= 2\pi rh$

volume $= \pi r^2 h$



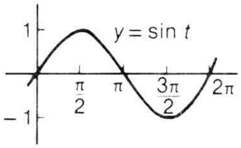
TRIGONOMETRIC FUNCTIONS

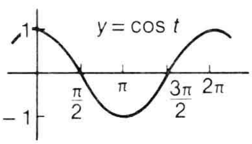


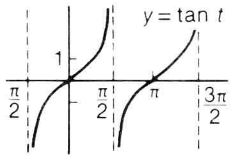


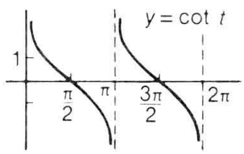
$$\sin t = \sin\theta = y = \frac{b}{r} \qquad \cos t = \cos\theta = x = \frac{a}{r}$$

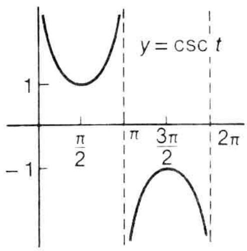
$$\tan t = \tan\theta = \frac{y}{x} = \frac{b}{a} \qquad \cot t = \cot\theta = \frac{x}{y} = \frac{a}{b}$$

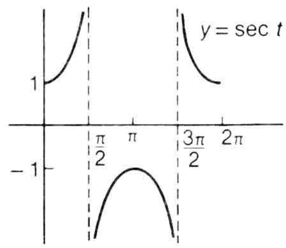












FORMULAS FROM TRIGONOMETRY

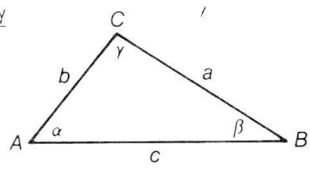
Angle Measures

$180° = \pi$ radians $\qquad x \text{ deg} = \left(x \cdot \frac{\pi}{180}\right) \text{ rad} \qquad x \text{ rad} = \left(x \cdot \frac{180}{\pi}\right) \text{ deg}$

Triangles

Law of sines

$$\frac{\sin\alpha}{a} = \frac{\sin\beta}{b} = \frac{\sin\gamma}{c}$$



Law of cosines

$$a^2 = b^2 + c^2 - 2bc \sin\alpha$$

$$b^2 = a^2 + c^2 - 2ac \sin\beta$$

$$c^2 = a^2 + b^2 - 2ab \sin\gamma$$

area: $A = \frac{1}{2}ab \sin\gamma = \frac{1}{2}bc \sin\alpha = \frac{1}{2}ac \sin\beta$

$A = \sqrt{s(s-a)(s-b)(s-c)}$ where $s = \frac{a+b+c}{2}$

TRIGONOMETRIC FUNCTION IDENTITIES

Basic Identities

$\sin x = \dfrac{1}{\csc x}$ $\quad$ $\cos x = \dfrac{1}{\sec x}$ $\quad$ $\tan x = \dfrac{1}{\cot x}$

$\tan x = \dfrac{\sin x}{\cos x}$ $\quad$ $\cot x = \dfrac{\cos x}{\sin x}$

$\sin^2 x + \cos^2 x = 1$ $\quad$ $1 + \tan^2 x = \sec^2 x$ $\quad$ $1 + \cot^2 x = \csc^2 x$

Odd–Even and Cofunction Identities

$\sin(-x) = -\sin x$ $\quad$ $\sin\left(\dfrac{\pi}{2} - x\right) = \cos x$ $\quad$ $\sin\left(\dfrac{\pi}{2} + x\right) = \cos x$

$\cos(-x) = \cos x$ $\quad$ $\cos\left(\dfrac{\pi}{2} - x\right) = \sin x$ $\quad$ $\cos\left(\dfrac{\pi}{2} + x\right) = -\sin x$

$\tan(-x) = -\tan x$ $\quad$ $\tan\left(\dfrac{\pi}{2} - x\right) = \cot x$ $\quad$ $\tan\left(\dfrac{\pi}{2} + x\right) = -\cot x$

Sum–Difference Identities

$\sin(x + y) = \sin x \cos y + \cos x \sin y$

$\cos(x + y) = \cos x \cos y - \sin x \sin y$

$\tan(x + y) = \dfrac{\tan x + \tan y}{1 - \tan x \tan y}$

$\sin(x - y) = \sin x \cos y - \cos x \sin y$

$\cos(x - y) = \cos x \cos y + \sin x \sin y$

$\tan(x - y) = \dfrac{\tan x - \tan y}{1 + \tan x \tan y}$

Double Angle Identities

$\sin 2x = 2 \sin x \cos x$

$\cos 2x = \cos^2 x - \sin^2 x$

$\tan 2x = \dfrac{2 \tan x}{1 - \tan^2 x}$

Half-Angle Identities

$\sin \dfrac{x}{2} = \pm\sqrt{\dfrac{1 - \cos x}{2}}$

$\cos \dfrac{x}{2} = \pm\sqrt{\dfrac{1 + \cos x}{2}}$

(sign determined by quadrant of $\dfrac{x}{2}$)

$\tan \dfrac{x}{2} = \dfrac{\sin x}{1 + \cos x} = \dfrac{1 - \cos x}{\sin x}$

Product to Sum Identities

$\sin x \sin y = \dfrac{1}{2}[\cos(x - y) - \cos(x + y)]$

$\cos x \cos y = \dfrac{1}{2}[\cos(x + y) + \cos(x - y)]$

$\sin x \cos y = \dfrac{1}{2}[\sin(x + y) + \sin(x - y)]$

Sum to Product Identities

$\sin x \pm \sin y = 2 \sin\left(\dfrac{x \pm y}{2}\right) \cos\left(\dfrac{x \mp y}{2}\right)$

$\cos x + \cos y = 2 \cos\left(\dfrac{x + y}{2}\right) \cos\left(\dfrac{x - y}{2}\right)$

$\cos x - \cos y = -2 \sin\left(\dfrac{x + y}{2}\right) \sin\left(\dfrac{x - y}{2}\right)$

TRIGONOMETRY

Preface

In revising the book *Trigonometry Using Calculators,* we felt a need for a title that more adequately describes the contents and philosophy of the book. The title that we liked initially was *Trigonometry: A Modern Approach,* but one reviewer pointed out that "every author since Hipparchus has used that title." After considerable discussion with colleagues, potential users, and publishers, we decided to follow an old and honorable tradition in naming the text.

The primary reason for our choice is that we believe this book really does offer a modern approach to trigonometry. Trigonometry has changed in fundamental ways in recent years, both *in terms of the applications,* which are of greatest importance to users, and *in terms of computational tools,* which affect the way trigonometry is learned and the convenience with which it is applied. This book reflects these changes, emphasizing the significance of the trigonometric functions as usable functions and utilizing part of the incredible technological development of high-speed computing devices as an integral part of the learning process.

Functional Trigonometry and Modern Technology

The most significant applications of trigonometry make use of the six functions, which were originally defined in terms of ratios of sides of triangles but which are now properly considered as functions of real numbers. As the world of science broadens, as more and more disciplines require mathematical treatment, more students need to understand more mathematics. Virtually all cyclic or periodic phenomena are best described by use of trigonometric functions. Students studying engineering and physics have always needed a functional treatment of trigonometry, but the need is now expanding to economics and natural resources, to all areas of business, and to many of the other social sciences.

Use of Calculators

Hand-held scientific calculators have completely changed the way we think about much of the world. They have become *the* tool needed by today's practicing scientists, engineers, and businesspeople. The slide rule, once the trademark of the engineer and a part of the core curriculum of every school in the country, is no longer even part of the vocabulary of engineers. Most engineering students don't know what a slide rule is, but few engineers today could survive for long without a calculator.

The calculator has altered trigonometry in just as fundamental a fashion. As is pointed out in Section 2.2, laboriously computed tables dating from the second century A.D. served all of science until the 1600s when more laborious work provided more detailed tables. The invention of calculus did not change our dependence upon tables. All of the computations performed with these tabular data needed logarithms to make them feasible, and manipulation of logarithms needed either more tables or a mechanical device such as a slide rule. The slide rule was incomparably handier but was severely limited in accuracy. *Now automatic computing devices are as handy as the slide rule and more accurate than the most elaborate tables.* It seems ridiculous to ignore what these remarkable computing tools have made possible.

Trigonometry texts have traditionally devoted a substantial amount of space to logarithms and linear interpolation in connection with extensive tables. Our feeling is that linear interpolation is little needed in a modern trigonometry course. We prefer that students who need it learn specific techniques for the tabular data with which they may have to work. Nonetheless, for those instructors who wish to treat interpolation in trigonometry, we have included a brief discussion in Appendix C. The appendixes include a substantial body of information, including numerous problem sets, and should be used as an important resource.

Looking Ahead to Computers

As technology continues to change, we must recognize that the calculator will not always remain what it is today. We attempt to anticipate some of the possible changes with two features in this book, a discussion of *hand-held computers* and *optional computer problems.*

Computers tend to become smaller, faster, and more powerful as they become more readily available. Several manufacturers are producing hand-held computers with the capabilities of microcomputers, including high-level language programming and even printers. In several footnotes throughout the text we address the use of such computers for needed computation. A brief discussion on their use is included in Appendix A.

Most chapters contain computer problems *for enrichment only*. These are not an essential part of the course, but are available for use when appropriate. See the next section for more information about the computer problems.

Basic Philosophy and Special Features of the Text

A number of features of this revision are significant. Topics have been reorganized and the book has been completely rewritten. A major goal has been to make all of the exposition readable by the student. Obviously every elementary text is intended for the student, but some are conspicuously more successful than others. We believe that student response will confirm that students can truly read and learn from the text.

The text is designed to:

- present and discuss ideas,
- illustrate the ideas by several examples worked in detail,
- give practice with carefully designed problem sets.

To accomplish these aims there are a number of distinctive features that should be noted.

Examples and Exercises

In the discussion of example problems, we explain why particular strategies are useful, and we point out both alternative approaches and pitfalls to be avoided. The problem sets anticipate later work and stimulate related ideas as well as providing the drill necessary to master skills and ideas. Each exercise set includes a variety of problems ranging from simple to challenging. Hints are given for some nonroutine problems.

Introductory Chapter Sections

Key ideas needed for the chapter are included in introductory review sections (with exercises). Instructors may use as much or as little of this material as needed for a specific class.

Chapter Summaries

Important ideas and theorems are summarized with the key formulas from the chapter to help students put their information into a more useful conceptual framework and to serve as a convenient reference.

In the summaries, and throughout the book, students are given an indication of uses and reasons for studying particular topics.

Review Exercises

An integrated set of exercises at the end of each chapter allows students to check mastery of concepts and pull together key ideas.

Optional Computer Problems

As noted above, these enrichment sections are included as a service to students and teachers. We recognize that most trigonometry courses do not have the leisure to permit an instructor to teach programming, nor do we expect programming experience from our students. An increasingly common feature of our schools, however, is that *many of our students are already familiar with computers* and that *many classrooms have computing facilities that are not well-used for the learning of mathematics*. The computer problems allow instructors to encourage students, either by themselves or with a teacher's guidance, to take advantage of a wide variety of problems, to develop computer skills, and more fully to understand the fundamental concepts of the chapter.

Exact and Approximate Calculations

Students are given practice with both *exact* (symbolic) and *approximate* (calculator) numbers and notation. Expressing answers in exact form involves application of definitions and/or basic concepts; giving results in decimal form provides familiarity with numbers as they occur in real-life settings. In this regard, we help students understand the meaning of significant digits in relation to the display shown on their calculators. The fact that the calculator shows eight or ten digits does not automatically invest those figures with meaning. A discussion of significant digits and their relation to the real and ideal worlds is included in Appendix B and in the text where needed.

Degree and Radian Modes

The fact that calculators deal differently with trigonometric functions in radian and degree mode is recognized and explained. For instance, there are two different sine functions as well as two inverse sine functions, depending on the mode. Both are shown to be useful.

Graphing

Graphs are emphasized for their great utility. Many of the crucial features of a function become obvious from a reasonable graph. The trigonometric functions are graphed in Chapter 3 and in Chapter 6, which we devote entirely to graphs. In this chapter we introduce new unifying ideas, *fundamental intervals and cycles*, which make the graphing of general trigonometric functions much simpler. Then the student is shown how to use graphs to help solve equations, revisiting ideas from previous chapters.

Organization

The organizational principle of revisiting ideas discussed in earlier chapters is at work throughout the book. The first chapter pulls together concepts needed for the whole book. Ideas introduced in one chapter are not dropped. We feel that the most effective mathematical learning takes place when a student revisits concepts, especially in a different setting. Thus inverse trigonometric functions are moved to Chapter 3. In general, the topic of inverse functions is difficult for students. In most books students are just introduced to these strange new functions, which are not encountered again until calculus. Here we review algebraic concepts, which place inverse trigonometric functions in a more familiar setting and then *use* them consistently throughout the remainder of the book. Similarly, identities are related to the more friendly algebraic identities and are then used to simplify the solution of equations and to make complex graphing easier. We identify and give students practice with key identities, which are crucial in so many subsequent courses.

Solving Triangles

The traditional approach was to formulate solutions to allow the use of logarithms for computations. We are freed from such constraints by the power of the calculator. Thus the Law of Cosines can be used more readily when it can tell us something about the nature of solutions. The ambiguous case receives separate treatment, utilizing the better mathematical tools students have developed. The numbers used in application problems can also be more realistic, since the calculator handles them as easily as contrived simple numbers.

Appendixes

Appendix A contains a relatively complete introduction to the use of calculators for those students who have had no previous experience with them. Both AOS calculators (based on algebraic entry) and RPN (Reverse Polish Notation) calculators are discussed. In most cases, the student can master this material without assistance. Further instruction on special function keys is given throughout the text as needed and when appropriate. A special addition discusses the use of hand-held computers. **Appendix B** includes a relatively detailed treatment of computation with approximate numbers, significant digits, and the significance of digits. **Appendix C** treats tables and linear interpolation. Each appendix includes **exercise sets.**

The book is designed for a one-semester or one-quarter course in trigonometry. Prerequisites of high school geometry and intermediate algebra are assumed.

Acknowledgments

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J.E.
Logan, Utah *C.J.E.*
December 1984 *L.O.C.*

TRIGONOMETRY

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