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J.A. Bondy
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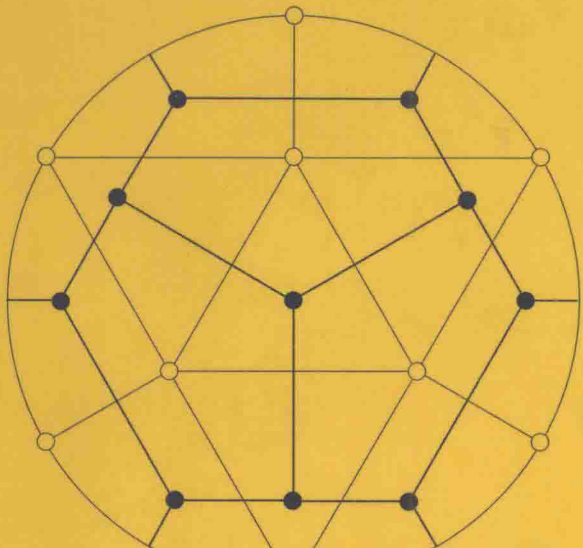
Graph Theory



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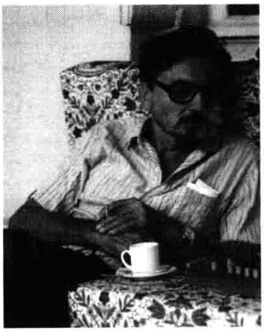
Dedication

To the memory of our dear friends and mentors



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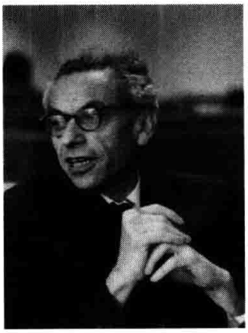

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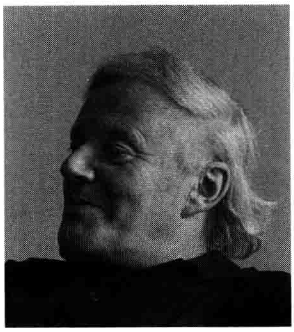

BILL TUTTE

Preface

For more than one hundred years, the development of graph theory was inspired and guided mainly by the Four-Colour Conjecture. The resolution of the conjecture by K. Appel and W. Haken in 1976, the year in which our first book *Graph Theory with Applications* appeared, marked a turning point in its history. Since then, the subject has experienced explosive growth, due in large measure to its role as an essential structure underpinning modern applied mathematics. Computer science and combinatorial optimization, in particular, draw upon and contribute to the development of the theory of graphs. Moreover, in a world where communication is of prime importance, the versatility of graphs makes them indispensable tools in the design and analysis of communication networks.

Building on the foundations laid by Claude Berge, Paul Erdős, Bill Tutte, and others, a new generation of graph-theorists has enriched and transformed the subject by developing powerful new techniques, many borrowed from other areas of mathematics. These have led, in particular, to the resolution of several longstanding conjectures, including Berge's Strong Perfect Graph Conjecture and Kneser's Conjecture, both on colourings, and Gallai's Conjecture on cycle coverings.

One of the dramatic developments over the past thirty years has been the creation of the theory of graph minors by G. N. Robertson and P. D. Seymour. In a long series of deep papers, they have revolutionized graph theory by introducing an original and incisive way of viewing graphical structure. Developed to attack a celebrated conjecture of K. Wagner, their theory gives increased prominence to embeddings of graphs in surfaces. It has led also to polynomial-time algorithms for solving a variety of hitherto intractable problems, such as that of finding a collection of pairwise-disjoint paths between prescribed pairs of vertices.

A technique which has met with spectacular success is the probabilistic method. Introduced in the 1940s by Erdős, in association with fellow Hungarians A. Rényi and P. Turán, this powerful yet versatile tool is being employed with ever-increasing frequency and sophistication to establish the existence or nonexistence of graphs, and other combinatorial structures, with specified properties.

As remarked above, the growth of graph theory has been due in large measure to its essential role in the applied sciences. In particular, the quest for efficient algorithms has fuelled much research into the structure of graphs. The importance of spanning trees of various special types, such as breadth-first and depth-first trees, has become evident, and tree decompositions of graphs are a central ingredient in the theory of graph minors. Algorithmic graph theory borrows tools from a number of disciplines, including geometry and probability theory. The discovery by S. Cook in the early 1970s of the existence of the extensive class of seemingly intractable $\mathcal{NP}$-complete problems has led to the search for efficient approximation algorithms, the goal being to obtain a good approximation to the true value. Here again, probabilistic methods prove to be indispensable.

The links between graph theory and other branches of mathematics are becoming increasingly strong, an indication of the growing maturity of the subject. We have already noted certain connections with topology, geometry, and probability. Algebraic, analytic, and number-theoretic tools are also being employed to considerable effect. Conversely, graph-theoretical methods are being applied more and more in other areas of mathematics. A notable example is Szemerédi's regularity lemma. Developed to solve a conjecture of Erdős and Turán, it has become an essential tool in additive number theory, as well as in extremal conbinatorics. An extensive account of this interplay can be found in the two-volume *Handbook of Combinatorics*.

It should be evident from the above remarks that graph theory is a flourishing discipline. It contains a body of beautiful and powerful theorems of wide applicability. The remarkable growth of the subject is reflected in the wealth of books and monographs now available. In addition to the *Handbook of Combinatorics*, much of which is devoted to graph theory, and the three-volume treatise on combinatorial optimization by Schrijver (2003), destined to become a classic, one can find monographs on colouring by Jensen and Toft (1995), on flows by Zhang (1997), on matching by Lovász and Plummer (1986), on extremal graph theory by Bollobás (1978), on random graphs by Bollobás (2001) and Janson et al. (2000), on probabilistic methods by Alon and Spencer (2000) and Molloy and Reed (1998), on topological graph theory by Mohar and Thomassen (2001), on algebraic graph theory by Biggs (1993), and on digraphs by Bang-Jensen and Gutin (2001), as well as a good choice of textbooks. Another sign is the significant number of new journals dedicated to graph theory.

The present project began with the intention of simply making minor revisions to our earlier book. However, we soon came to the realization that the changing face of the subject called for a total reorganization and enhancement of its contents. As with *Graph Theory with Applications*, our primary aim here is to present a coherent introduction to the subject, suitable as a textbook for advanced undergraduate and beginning graduate students in mathematics and computer science. For pedagogical reasons, we have concentrated on topics which can be covered satisfactorily in a course. The most conspicuous omission is the theory of graph minors, which we only touch upon, it being too complex to be accorded an adequate treatment. We have maintained as far as possible the terminology and notation of our earlier book, which are now generally accepted.

Particular care has been taken to provide a systematic treatment of the theory of graphs without sacrificing its intuitive and aesthetic appeal. Commonly used proof techniques are described and illustrated. Many of these are to be found in insets, whereas others, such as search trees, network flows, the regularity lemma and the local lemma, are the topics of entire sections or chapters. The exercises, of varying levels of difficulty, have been designed so as to help the reader master these techniques and to reinforce his or her grasp of the material. Those exercises which are needed for an understanding of the text are indicated by a star. The more challenging exercises are separated from the easier ones by a dividing line.

A second objective of the book is to serve as an introduction to research in graph theory. To this end, sections on more advanced topics are included, and a number of interesting and challenging open problems are highlighted and discussed in some detail. These and many more are listed in an appendix.

Despite this more advanced material, the book has been organized in such a way that an introductory course on graph theory may be based on the first few sections of selected chapters. Like number theory, graph theory is conceptually simple, yet gives rise to challenging unsolved problems. Like geometry, it is visually pleasing. These two aspects, along with its diverse applications, make graph theory an ideal subject for inclusion in mathematical curricula.

We have sought to convey the aesthetic appeal of graph theory by illustrating the text with many interesting graphs — a full list can be found in the index. The cover design, taken from Chapter 10, depicts simultaneous embeddings on the projective plane of K_6 and its dual, the Petersen graph.

A Web page for the book is available at

`http://blogs.springer.com/bondyandmurty`

The reader will find there hints to selected exercises, background to open problems, other supplementary material, and an inevitable list of errata. For instructors wishing to use the book as the basis for a course, suggestions are provided as to an appropriate selection of topics, depending on the intended audience.

We are indebted to many friends and colleagues for their interest in and help with this project. Tommy Jensen deserves a special word of thanks. He read through the entire manuscript, provided numerous unfailingly pertinent comments, simplified and clarified several proofs, corrected many technical errors and linguistic infelicities, and made valuable suggestions. Others who went through and commented on parts of the book include Noga Alon, Roland Assous, Xavier Buchwalder, Genghua Fan, Frédéric Havet, Bill Jackson, Stephen Locke, Zsolt Tuza, and two anonymous readers. We were most fortunate to benefit in this way from their excellent knowledge and taste.

Colleagues who offered advice or supplied exercises, problems, and other helpful material include Michael Albertson, Marcelo de Carvalho, Joseph Cheriyan, Roger Entringer, Herbert Fleischner, Richard Gibbs, Luis Goddyn, Alexander

Kelmans, Henry Kierstead, László Lovász, Cláudio Lucchesi, George Purdy, Dieter Rautenbach, Bruce Reed, Bruce Richmond, Neil Robertson, Alexander Schrijver, Paul Seymour, Miklós Simonovits, Balazs Szegedy, Robin Thomas, Stéphan Thomassé, Carsten Thomassen, and Jacques Verstraëte. We thank them all warmly for their various contributions. We are grateful also to Martin Crossley for allowing us to use (in Figure 10.24) drawings of the Möbius band and the torus taken from his book Crossley (2005).

Facilities and support were kindly provided by Maurice Pouzet at Université Lyon 1 and Jean Fonlupt at Université Paris 6. The glossary was prepared using software designed by Nicola Talbot of the University of East Anglia. Her promptly-offered advice is much appreciated. Finally, we benefitted from a fruitful relationship with Karen Borthwick at Springer, and from the technical help provided by her colleagues Brian Bishop and Frank Ganz.

We are dedicating this book to the memory of our friends Claude Berge, Paul Erdős, and Bill Tutte. It owes its existence to their achievements, their guiding hands, and their personal kindness.

J.A. Bondy and U.S.R. Murty

September 2007

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Graphs

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1.1 Graphs and Their Representation

Definitions and Examples

Many real-world situations can conveniently be described by means of a diagram consisting of a set of points together with lines joining certain pairs of these points.

For example, the points could represent people, with lines joining pairs of friends; or the points might be communication centres, with lines representing communication links. Notice that in such diagrams one is mainly interested in whether two given points are joined by a line; the manner in which they are joined is immaterial. A mathematical abstraction of situations of this type gives rise to the concept of a graph.

A *graph* G is an ordered pair $(V(G), E(G))$ consisting of a set $V(G)$ of *vertices* and a set $E(G)$, disjoint from $V(G)$, of *edges*, together with an *incidence function* ψ_G that associates with each edge of G an unordered pair of (not necessarily distinct) vertices of G. If e is an edge and u and v are vertices such that $\psi_G(e) = \{u, v\}$, then e is said to *join* u and v, and the vertices u and v are called the *ends* of e. We denote the numbers of vertices and edges in G by $v(G)$ and $e(G)$; these two basic parameters are called the *order* and *size* of G, respectively.

Two examples of graphs should serve to clarify the definition. For notational simplicity, we write uv for the unordered pair $\{u, v\}$.

Example 1.

$$G = (V(G), E(G))$$

where

$$\begin{aligned} V(G) &= \{u, v, w, x, y\} \\ E(G) &= \{a, b, c, d, e, f, g, h\} \end{aligned}$$

and ψ_G is defined by

$$\begin{array}{llll} \psi_G(a) = uv & \psi_G(b) = uu & \psi_G(c) = vw & \psi_G(d) = wx \\ \psi_G(e) = vx & \psi_G(f) = wx & \psi_G(g) = ux & \psi_G(h) = xy \end{array}$$

Example 2.

$$H = (V(H), E(H))$$

where

$$\begin{aligned} V(H) &= \{v_0, v_1, v_2, v_3, v_4, v_5\} \\ E(H) &= \{e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8, e_9, e_{10}\} \end{aligned}$$

and ψ_H is defined by

$$\begin{array}{lllll} \psi_H(e_1) = v_1v_2 & \psi_H(e_2) = v_2v_3 & \psi_H(e_3) = v_3v_4 & \psi_H(e_4) = v_4v_5 & \psi_H(e_5) = v_5v_1 \\ \psi_H(e_6) = v_0v_1 & \psi_H(e_7) = v_0v_2 & \psi_H(e_8) = v_0v_3 & \psi_H(e_9) = v_0v_4 & \psi_H(e_{10}) = v_0v_5 \end{array}$$

Drawings of Graphs

Graphs are so named because they can be represented graphically, and it is this graphical representation which helps us understand many of their properties. Each vertex is indicated by a point, and each edge by a line joining the points representing its ends. Diagrams of G and H are shown in Figure 1.1. (For clarity, vertices are represented by small circles.)

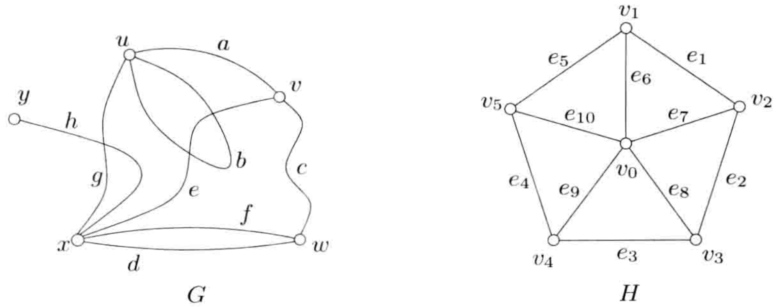


Fig. 1.1. Diagrams of the graphs G and H

There is no single correct way to draw a graph; the relative positions of points representing vertices and the shapes of lines representing edges usually have no significance. In Figure 1.1, the edges of G are depicted by curves, and those of H by straight-line segments. A diagram of a graph merely depicts the incidence relation holding between its vertices and edges. However, we often draw a diagram of a graph and refer to it as the graph itself; in the same spirit, we call its points 'vertices' and its lines 'edges'.

Most of the definitions and concepts in graph theory are suggested by this graphical representation. The ends of an edge are said to be *incident* with the edge, and *vice versa*. Two vertices which are incident with a common edge are *adjacent*, as are two edges which are incident with a common vertex, and two distinct adjacent vertices are *neighbours*. The set of neighbours of a vertex v in a graph G is denoted by $N_G(v)$.

An edge with identical ends is called a *loop*, and an edge with distinct ends a *link*. Two or more links with the same pair of ends are said to be *parallel edges*. In the graph G of Figure 1.1, the edge b is a loop, and all other edges are links; the edges d and f are parallel edges.

Throughout the book, the letter G denotes a graph. Moreover, when there is no scope for ambiguity, we omit the letter G from graph-theoretic symbols and write, for example, V and E instead of $V(G)$ and $E(G)$. In such instances, we denote the numbers of vertices and edges of G by n and m, respectively.

A graph is *finite* if both its vertex set and edge set are finite. In this book, we mainly study finite graphs, and the term 'graph' always means 'finite graph'. The graph with no vertices (and hence no edges) is the *null graph*. Any graph with just one vertex is referred to as *trivial*. All other graphs are *nontrivial*. We admit the null graph solely for mathematical convenience. Thus, unless otherwise specified, all graphs under discussion should be taken to be nonnull.

A graph is *simple* if it has no loops or parallel edges. The graph H in Example 2 is simple, whereas the graph G in Example 1 is not. Much of graph theory is concerned with the study of simple graphs.

A set V, together with a set E of two-element subsets of V, defines a simple graph (V, E), where the ends of an edge uv are precisely the vertices u and v. Indeed, in any simple graph we may dispense with the incidence function ψ by renaming each edge as the unordered pair of its ends. In a diagram of such a graph, the labels of the edges may then be omitted.

Special Families of Graphs

Certain types of graphs play prominent roles in graph theory. A *complete graph* is a simple graph in which any two vertices are adjacent, an *empty graph* one in which no two vertices are adjacent (that is, one whose edge set is empty). A graph is *bipartite* if its vertex set can be partitioned into two subsets X and Y so that every edge has one end in X and one end in Y; such a partition (X, Y) is called a *bipartition* of the graph, and X and Y its *parts*. We denote a bipartite graph G with bipartition (X, Y) by $G[X, Y]$. If $G[X, Y]$ is simple and every vertex in X is joined to every vertex in Y, then G is called a *complete bipartite graph*. A *star* is a complete bipartite graph $G[X, Y]$ with $|X| = 1$ or $|Y| = 1$. Figure 1.2 shows diagrams of a complete graph, a complete bipartite graph, and a star.

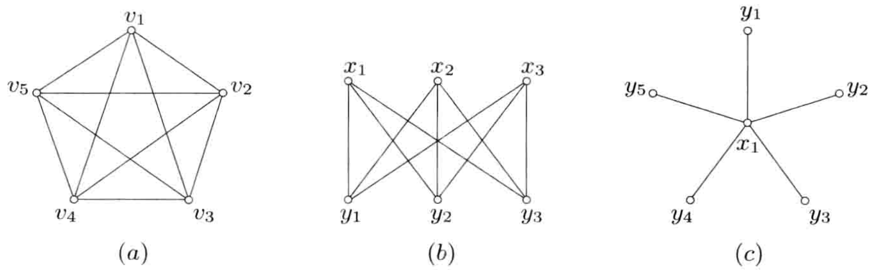


Fig. 1.2. (a) A complete graph, (b) a complete bipartite graph, and (c) a star

A *path* is a simple graph whose vertices can be arranged in a linear sequence in such a way that two vertices are adjacent if they are consecutive in the sequence, and are nonadjacent otherwise. Likewise, a *cycle* on three or more vertices is a simple graph whose vertices can be arranged in a cyclic sequence in such a way that two vertices are adjacent if they are consecutive in the sequence, and are nonadjacent otherwise; a cycle on one vertex consists of a single vertex with a loop, and a cycle on two vertices consists of two vertices joined by a pair of parallel edges. The *length* of a path or a cycle is the number of its edges. A path or cycle of length k is called a *k-path* or *k-cycle*, respectively; the path or cycle is *odd* or *even* according to the parity of k. A 3-cycle is often called a *triangle*, a 4-cycle a *quadrilateral*, a 5-cycle a *pentagon*, a 6-cycle a *hexagon*, and so on. Figure 1.3 depicts a 3-path and a 5-cycle.

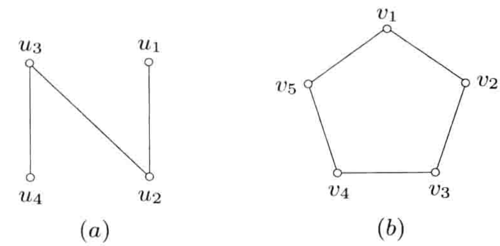


Fig. 1.3. (a) A path of length three, and (b) a cycle of length five

A graph is *connected* if, for every partition of its vertex set into two nonempty sets X and Y, there is an edge with one end in X and one end in Y; otherwise the graph is *disconnected*. In other words, a graph is disconnected if its vertex set can be partitioned into two nonempty subsets X and Y so that no edge has one end in X and one end in Y. (It is instructive to compare this definition with that of a bipartite graph.) Examples of connected and disconnected graphs are displayed in Figure 1.4.

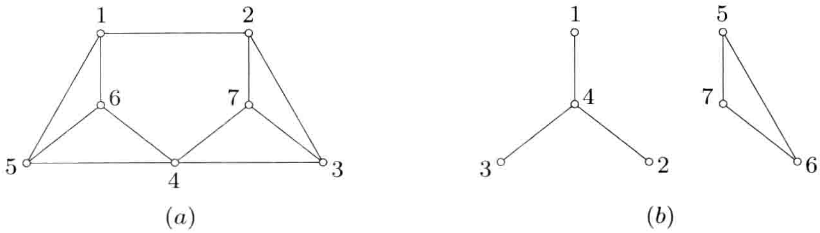


Fig. 1.4. (a) A connected graph, and (b) a disconnected graph

As observed earlier, examples of graphs abound in the real world. Graphs also arise naturally in the study of other mathematical structures such as polyhedra, lattices, and groups. These graphs are generally defined by means of an adjacency rule, prescribing which unordered pairs of vertices are edges and which are not. A number of such examples are given in the exercises at the end of this section and in Section 1.3.

For the sake of clarity, we observe certain conventions in representing graphs by diagrams: we do not allow an edge to intersect itself, nor let an edge pass through a vertex that is not an end of the edge; clearly, this is always possible. However, two edges may intersect at a point that does not correspond to a vertex, as in the drawings of the first two graphs in Figure 1.2. A graph which can be drawn in the plane in such a way that edges meet only at points corresponding to their common ends is called a *planar graph*, and such a drawing is called a *planar embedding* of the graph. For instance, the graphs G and H of Examples 1 and 2 are both planar, even though there are crossing edges in the particular drawing of G shown in Figure 1.1. The first two graphs in Figure 1.2, on the other hand, are not planar, as proved later.

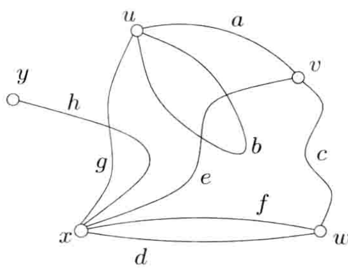
Although not all graphs are planar, every graph can be drawn on some surface so that its edges intersect only at their ends. Such a drawing is called an *embedding* of the graph on the surface. Figure 1.21 provides an example of an embedding of a graph on the torus. Embeddings of graphs on surfaces are discussed in Chapter 3 and, more thoroughly, in Chapter 10.

Incidence and Adjacency Matrices

Although drawings are a convenient means of specifying graphs, they are clearly not suitable for storing graphs in computers, or for applying mathematical methods to study their properties. For these purposes, we consider two matrices associated with a graph, its incidence matrix and its adjacency matrix.

Let G be a graph, with vertex set V and edge set E. The *incidence matrix* of G is the $n \times m$ matrix $\mathbf{M}_G := (m_{ve})$, where m_{ve} is the number of times (0, 1, or 2) that vertex v and edge e are incident. Clearly, the incidence matrix is just another way of specifying the graph.

The *adjacency matrix* of G is the $n \times n$ matrix $\mathbf{A}_G := (a_{uv})$, where a_{uv} is the number of edges joining vertices u and v, each loop counting as two edges. Incidence and adjacency matrices of the graph G of Figure 1.1 are shown in Figure 1.5.



G

	a	b	c	d	e	f	g	h
u	1	2	0	0	0	0	1	0
v	1	0	1	0	1	0	0	0
w	0	0	1	1	0	1	0	0
x	0	0	0	1	1	1	1	1
y	0	0	0	0	0	0	0	1

$\mathbf{M}$

	u	v	w	x	y
u	2	1	0	1	0
v	1	0	1	1	0
w	0	1	0	2	0
x	1	1	2	0	1
y	0	0	0	1	0

$\mathbf{A}$

Fig. 1.5. Incidence and adjacency matrices of a graph

Because most graphs have many more edges than vertices, the adjacency matrix of a graph is generally much smaller than its incidence matrix and thus requires less storage space. When dealing with simple graphs, an even more compact representation is possible. For each vertex v, the neighbours of v are listed in some order. A list $(N(v) : v \in V)$ of these lists is called an *adjacency list* of the graph. Simple graphs are usually stored in computers as adjacency lists.

When G is a bipartite graph, as there are no edges joining pairs of vertices belonging to the same part of its bipartition, a matrix of smaller size than the adjacency matrix may be used to record the numbers of edges joining pairs of vertices. Suppose that $G[X, Y]$ is a bipartite graph, where $X := \{x_1, x_2, \ldots, x_r\}$ and $Y := \{y_1, y_2, \ldots, y_s\}$. We define the *bipartite adjacency matrix* of G to be the $r \times s$ matrix $\mathbf{B}_G = (b_{ij})$, where b_{ij} is the number of edges joining x_i and y_j.

Vertex Degrees

The *degree* of a vertex v in a graph G, denoted by $d_G(v)$, is the number of edges of G incident with v, each loop counting as two edges. In particular, if G is a simple graph, $d_G(v)$ is the number of neighbours of v in G. A vertex of degree zero is called an *isolated vertex.* We denote by $\delta(G)$ and $\Delta(G)$ the minimum and maximum degrees of the vertices of G, and by $d(G)$ their *average degree*, $\frac{1}{n}\sum_{v \in V} d(v)$. The following theorem establishes a fundamental identity relating the degrees of the vertices of a graph and the number of its edges.

Theorem 1.1 *For any graph G,*

$$\sum_{v \in V} d(v) = 2m \tag{1.1}$$

Proof Consider the incidence matrix $\mathbf{M}$ of G. The sum of the entries in the row corresponding to vertex v is precisely $d(v)$. Therefore $\sum_{v \in V} d(v)$ is just the sum of all the entries in $\mathbf{M}$. But this sum is also $2m$, because each of the m column sums of $\mathbf{M}$ is 2, each edge having two ends. □

Corollary 1.2 *In any graph, the number of vertices of odd degree is even.*

Proof Consider equation (1.1) modulo 2. We have

$$d(v) \equiv \begin{cases} 1 \pmod 2 & \text{if } d(v) \text{ is odd,} \\ 0 \pmod 2 & \text{if } d(v) \text{ is even.} \end{cases}$$

Thus, modulo 2, the left-hand side is congruent to the number of vertices of odd degree, and the right-hand side is zero. The number of vertices of odd degree is therefore congruent to zero modulo 2. □

A graph G is *k-regular* if $d(v) = k$ for all $v \in V$; a *regular graph* is one that is k-regular for some k. For instance, the complete graph on n vertices is $(n-1)$-regular, and the complete bipartite graph with k vertices in each part is k-regular. For $k = 0, 1$ and 2, k-regular graphs have very simple structures and are easily characterized (Exercise 1.1.5). By contrast, 3-regular graphs can be remarkably complex. These graphs, also referred to as *cubic* graphs, play a prominent role in graph theory. We present a number of interesting examples of such graphs in the next section.