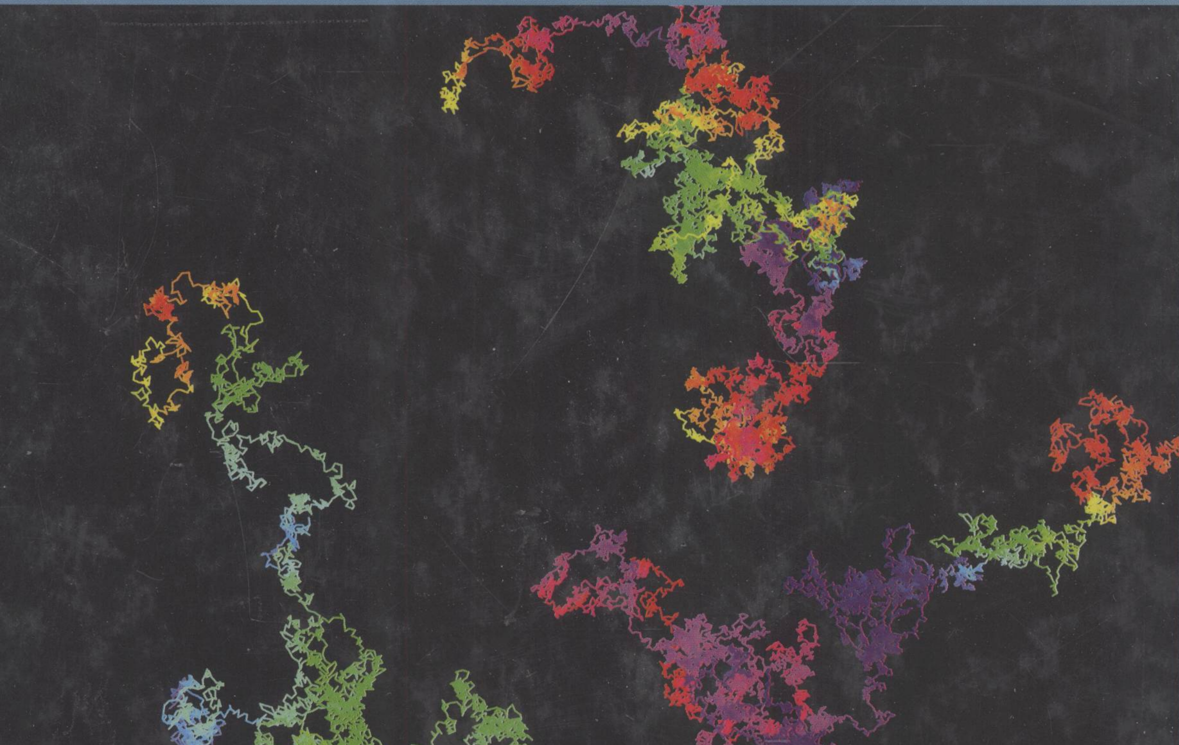
Elements of the Random Walk

An Introduction for Advanced Students and Researchers



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An Introduction for Advanced Students and Researchers

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ELEMENTS OF THE RANDOM WALK

An Introduction for Advanced Students and Researchers

Random walks have proven to be a useful model in understanding processes across a wide spectrum of scientific disciplines. *Elements of the Random Walk* is an introduction to some of the most powerful and general techniques used in the application of these ideas.

The mathematical construct that runs through the analysis of each of the topics covered in this book, and which therefore unifies the mathematical treatment, is the generating function. Although the reader is introduced to modern analytical tools, such as path integrals and field-theoretical formalism, the book is self-contained in that basic concepts are developed and relevant fundamental findings fully discussed. The book also provides an excellent introduction to frontier topics such as fractals, scaling and critical exponents, path integrals, application of the GLW Hamiltonian formalism, and renormalization group theory as they relate to the random walk problem. Mathematical background is provided in supplements at the end of each chapter, when appropriate.

This self-contained text will appeal to graduate students across science, engineering, and mathematics who need to understand the application of random walk techniques, as well as to established researchers.

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For Alice and Nancy

Preface

We begin this preface by reporting the results of an experiment. On April 23, 2003, we logged onto INSPEC – the physical science and engineering online literature service – and entered the phrase "random walk." In response to this query, INSPEC delivered a list of 5010 articles, published between 1967 and that date. We then tried the plural phrase, "random walks," and were informed of 1966 more papers. Some redundancy no doubt reduces the total number of references we received to a quantity less than the sum of those two figures. Nevertheless, the point has been made. Random walkers pervade science and technology.

Why is this so? Think of a system – by which we mean just about anything – that undergoes a series of relatively small changes and that does so at random. It is more likely than not that important aspects of this system's behavior can be understood in terms of the random walk. The canonical manifestation of the random walk is Brownian motion, the jittering of a small particle as it is knocked about by the molecules in a liquid or a gas. Chitons meandering on a sandy beach in search of food leave a random walker's trail, and the bacteria *E. coli* execute a random walk as they alternate between purposeful swimming and tumbling. Go to a casino, sit at the roulette wheel and see what kind of luck you have. The height of your pile of chips will follow the rules governing a random walk, although in this case the walk is biased (see Chapter 5), in that, statistically speaking, your collection of chips will inevitably shrink.

We could go on. Random walks play a role in the analysis of the movements of stock prices. *A Random Walk down Wall Street*, by Burton Malkiel has just been published completely revised, following eight previous editions. *Random Walks, Critical Phenomena, and Triviality in Quantum Field Theory*, by Roberto Fernandez *et al.* focuses on the behavior of quantum field theory in higher dimensions. There are also *Random Walks and Other Essays: Ruminations of a so-so Manager*, by René Azurin; *Random Walk: A Novel for a New Age*, by Lawrence Block, and the record *Random Walks Piano Music*, by David Kraehenbuehl and Martha

Braden. Which is to say, the idea of the random walk has seeped into our collective unconscious.

In this book, we hope to acquaint the reader with powerful techniques for the analysis of random walks. The book is intended for the interested student or researcher in physics, chemistry, engineering, or mathematics. It is our hope that the level, style, and content of the book will be appealing and useful to advanced undergraduate students, graduate students, and research scientists in all disciplines. The mathematical techniques used in developing the theory are either explained in the text proper or relegated to supplements at the end of each chapter when it was thought that their inclusion would interrupt the flow of the discussion. We are hopeful that a student with a good understanding of calculus ought to be able to follow much of the analytical manipulations. However, there are instances where more advanced mathematical familiarity would be helpful.

The first five chapters of this book focus on features of a variety of unrestricted walks – that is to say, the trails left by walkers that retain no memory of where they have visited previously – including biased walks, persistent walks, continuous time walks, continuous flow walks, and walks confined to restricted regions of space. The treatment is standard for the most part. However, we attempt to introduce a language and a point of view based on generating functions, which is consistent with a more modern field-theoretical approach to the subject. This method will be fully developed in the later chapters, when we must confront the complications introduced by requiring the walk to be self-avoiding, meaning that the walker's path can never intersect itself. The generating function not only provides for a field-theoretical representation of the walker's statistical behavior, but also allows for the connection to a statistical mechanical model of magnetism. The identification of random walks and magnetism has led to a quantum jump in our understanding of the effects of self-avoidance; it makes available to the theorist the full arsenal of analytical techniques that proved so successful in unraveling the complex properties of systems that undergo continuous phase transitions.

A brief overview of the subjects covered in this book is as follows. Chapter 1 begins with a discussion of the properties of a one-dimensional walk. The chapter is intended as a sort of overture, in that points of view and tricks are introduced that we develop more fully in later chapters. Chapter 2 contains a serious discussion of the meaning, nature, and implementation of the generating function in the context of the random walk. In Chapter 3, we utilize the generating function to investigate various aspects of unrestricted walks, including recurrence, mean number of sites visited, and first passage times. Chapter 4 – which relies heavily on the wonderful little book *Random Walks in Biology*, by Berg, and contains discussions of the effects of boundary conditions on walks – introduces the electrostatic analogy for the analysis of a walk in the steady state. Biased and persistent walks make their appearance in Chapter 5. We generalize the method of treating persistent walks in one dimension to higher dimensional walks and present complete solutions for persistent walks in two and three spatial dimensions. Chapter 6 is devoted entirely to the problem of characterizing the average shape of the trail left by a random walker. We focus on a particular quantitative measure of the shape of an object that is unusually well suited to the kind of analytical tools that now exist for the characterization of the properties of a random walk.

It should be mentioned that, in each of these chapters, we attempt to point out the usefulness of the concepts of models of actual physical and biological processes as the subject is developed. No attempt is made at a comprehensive comparison between predictions of the model and experiments on particular systems. We direct the reader to Weiss' book *Aspects and Applications of the Random Walk* for such detailed comparisons.

The random walk is one of the most important and intuitively appealing examples of a statistical field theory. It is a useful pedagogical model with which to introduce someone to the latest techniques of such a theory, such as Ginzburg–Landau–Wilson effective Hamiltonians, renormalization group theory, and graphical techniques. Finding and understanding the original literature, particularly when one is branching out beyond his or her field of specialty, can be a daunting task. We have tried to reorganize and synthesize the most recent advances in the subject, which in many cases are quite formidable in formulation. In so doing, we intended to make these theories of random walks accessible to those who will find the model useful but are not well versed in the mathematical techniques upon which many recent theoretical developments are based. We set out to accomplish this task in Chapters 7 through 12.

A reading of the table of contents clearly indicates what each of these chapters entails. Here we only point out a few of the features which we found to be particularly interesting. In Chapter 7 we embark on a field theory formulation of the random walk problem *à la* S.F. Edwards, by establishing a path integral expression for the generating function. Once this is accomplished, it is straightforward to generate a perturbation expansion in a quantity which measures "self-avoidance." Doing this allows for a gentle introduction to Feynman-like graphs and an exposition of the associated graphical algebraic techniques. Using rather simple scaling arguments, we clearly demonstrate the crucial role played by dimensionality in determining the behavior of the walker, a feature that is stressed throughout the book. Finally, a mean field theory of self-avoidance is identified which then permits the infinite perturbation series to be summed. The mean field generating function yields an expression for statistical properties of the walk which shows it is exactly equivalent to Flory's treatment of self-avoidance. Chapter 8 contains a brief review of general scaling notions as they apply to the random walk.

In Chapter 9, we establish the connection between the generating function and the correlation function of a fictional magnetic system, the $O(n)$ model. This is an extremely important result, for it brings to the theorist a new set of mathematical tools developed over years by statistical physicists in their study of critical phenomena, which can now be applied to the random walk problem. Critical point scaling, critical exponents, universality, effective Hamiltonians, and renormalization group theory are now at our disposal. These topics are covered in the remaining chapters.

Once the connection between magnetism and random walks has been established, mean field theory and its extensions can be studied in well-known ways. This is done in Chapters 9 and 10. The mean field result is shown to be identical to that found previously, thereby independently demonstrating the correctness of the $O(n)$ representation of random walks. Fluctuations are incorporated in a spin wave approximation, leading to a reasonable physical rendering of the condensed state of the magnetic system as it relates to the random walker. We outline the conceptual underpinnings of the renormalization group approach and present some simple realizations of the method in Chapter 11. Chapter 12 contains a full treatment of the renormalization group as it applies to self-avoiding random walks.

We have interspersed problems throughout each chapter. These are intended to be an aid in understanding the material and to provide a way for the reader to participate in the exploration of the subject. They were not designed to be excessively long or difficult. We suggest students attempt their solution as they work their way through the chapter as a way of gauging their understanding of the material. This book is intended to be a textbook, appropriate for a stand-alone course on random walks or as a supplemental text in a field in which an understanding of random walks is required. For example, this text might prove useful in a course on polymers, or one on advanced topics in statistical mechanics, or even quantum field theory. Since our purpose here is to create a textbook, we have decided not to encumber the presentation with a plethora of footnotes and an associated comprehensive bibliography. It is our hope that the references we have included can be used to track down the original articles dealing with the various aspects of the book. We apologize to all those researchers who have made major contributions to the field and whose work is not cited herein.

This book took shape over several years, and the authors have benefited from the contributions of a number of people. We would like to express our gratitude to Professor Fereydoon Family, who stimulated our initial interest in the subject of random walks, and to Arezki Beldjenna whose contributions to the joint research that underlies much of our chapter on shapes were especially important. We are grateful to the students who sat in on the graduate seminar on random walks at

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The possibility of our writing a book on random walks was initially raised by Professor Lui Lam. Our decision to publish with Cambridge University Press arose from discussions with Rufus Neal. We thank him for brokering what has turned out to be an enjoyable relationship with CUP, and for introducing us to Simon Capelin, who has proven to be everything we could want in an editor. We thank Fiona Chapman for her careful, and most cheerful, efforts as copy editor. We are also grateful to Professor Warren Esty for permission to reproduce the images used in Figures 1.1 and 1.2.

Finally, we are especially indebted to Professor Peter Young, who carefully read the next-to-final version of this manuscript. His queries, comments, and suggestions resulted in a greatly improved final version.

One of the authors (GG) expresses his appreciation to Tara and Bami Das for making the early years among the best. He is also indebted to Nancy for her loving support from the beginning to the end of this project. The other author (JR) thanks his wife Alice for support, love, advice, and forbearance.

Contents

1
Introduction to techniques

This entire book is, in one way or another, devoted to a single process: the random walk. As we will see, the rules that control the random walk are simple, even when we add elaborations that turn out to have considerable significance. However, as often occurs in mathematics and the physical sciences, the consequences of simple rules are far from elementary. We will also discover that random walks, as interesting as they are in themselves, provide a basis for the understanding of a wide range of phenomena. This is true in part because random walk processes are relevant to so many processes in such a wide range of contexts. It also follows from the fact that the solution of the random walk problem requires the use of so many of the mathematical techniques that have been developed and applied in contemporary twentieth-century physics. We'll start out simply, but it won't be long before we enounter aspects to the problem that invite – indeed require – intense scrutiny.

We begin our investigations by looking at the random walk in its most elementary manifestation. The reader may find that most of what follows in this chapter is familiar material. It is, nevertheless, useful to read through it. For one thing, review is always helpful. More importantly, connections that are hinted at in the early portions of this book will play an important role in later discussion.

1.1 The simplest walk

In the simplest example of a random walk the walker is confined to a straight line. This kind of walk is called, appropriately enough, a one-dimensional walk. In this case, steps take the walker in one direction or the other. We will call those two directions "right" and "left." This makes everything easy, as we can now describe the location of the walker by drawing a horizontal line on the page and showing where on the line the walker happens to be. Let's imagine that the walker decides where its next step takes it by flipping a coin. If the coin falls heads up the walker takes a step to the right; if the coin falls tails up the walker takes a step to the left.

The outcome of a flip of the coin is equally likely to be heads or tails, so the walk is clearly unbiased, in that there is no preference for progress to the left or the right.

Suppose the walker has taken N steps. It will have flipped the coin N times. If there were n heads and $N - n$ tails, the walker will have taken n steps to the right and $N - n$ steps to the left. Suppose that each step is l meters long. Then the walker will have moved a distance

$$\begin{aligned} d &= nl - (N - n)l \\ &= l(2n - N) \end{aligned} \tag{1.1}$$

to the right. The walker will thus end up Nl meters to the left of where it started, Nl meters to the right, or somewhere in between.

Before proceeding with the analysis of the behavior of the one-dimensional walker, it is useful to inquire as to the relevance of the notion of such a walker to the real world. As it turns out, the one-dimensional walk models a number of interesting physical and mathematical processes. There is, for example, the diffusive spreading, in one dimension, of a group of molecules or small particles as the result of thermal motion. The one-dimensional walk also represents an idealization of a chain-like polymer whose monomeric units can take on one of two possible conformations. The outcome of a simple game of chance – for instance, one governed by the flip of a coin – can also be described in terms of the eventual location of a one-dimensional random walker. In this last context, one of the first applications of notions eventually associated with the random walk is due to the mathematician de Moivre in the solution of the "gambler's ruin" problem (Montroll and Shlesinger, 1983).

An immediate and fairly obvious question about the walker is the sort one generally asks about the outcome of a random process, and that is with what probability the walker ends up at a given location. That question is equivalent to asking with what probability the walker throws a certain number of heads and tails in N tosses of the coin. Another way to visualize this problem is to consider the act of flipping a coin a "trial" and to call all flips that lead to heads a success. Then, clearly, the above probability is the same as the probability of obtaining n successes in N trials. Note that this interpretation applies to trials with more than two outcomes.

Back to the random walker. Suppose we want to know the probability that the walker has gone a distance d to the right of its original position. In terms of the net distance traveled, $d = l(2n - N)$, the number of heads that were thrown is given by

$$n = \frac{1}{2}\left(\frac{d}{l} + N\right) \tag{1.2}$$

Fig. 1.1. A particular outcome of three flips of a Roman coin displaying an image of Emperor Septimius Severus (AD 193–211). Shown, left to right, is a head, then a tail, then a head.

and the number of tails is

$$N - n = \frac{1}{2}\left(N - \frac{d}{l}\right) \tag{1.3}$$

Now, the probability of throwing a specific sequence that consists of n heads and $N - n$ tails in N coin tosses is equal to $(1/2)^N$. See Figure 1.1. We arrive at the result $(1/2)^N$ for this probability by noting that the probability of either result is one half. Specifying the exact sequence of heads and tails is the same as specifying the sequence of outcomes in a set of N trials, each of which has two possible results. To obtain the probability of this sequence of outcomes, we multiply together the probabilities of each outcome in the sequence. We obtain the probability in this way because each toss of the coin is statistically independent of all other coin flips. That is, the probability of a given flip yielding a head is 1/2, regardless of how all previous tosses turned out.

The probability of throwing n heads and $N - n$ tails in *any* order is $(1/2)^N$ multiplied by the number of sequences of n heads and $N - n$ tails. See, for example, Figure 1.2. This number is simply the binomial coefficient:

$$\binom{N}{n} = \frac{N!}{n!(N-n)!}. \tag{1.4}$$

To derive the combinatorial factor in (1.4) in the case of the coin flips depicted in Figures 1.1 and 1.2, imagine the sequence of flips in Figure 1.1 as an array of coins. Then shuffle the coins in all possible ways. There are $3 \times 2 \times 1 = 3!$ ways of doing this (three possibilities for the leftmost coin, two for the next in line and only one left to place at the far right). However, in shuffling the coins, you have overcounted the number of ways in which heads and tails can turn out. Switching the first and third coins in Figure 1.1 does not change the sequence of heads and tails as both are heads. To compensate for this overcounting, we divide 3! by 2!, the number of ways of shuffling, or permuting, the two heads. This leaves us with three

1
2
3

Fig. 1.2. All outcomes of three flips of the Roman coin in Figure 1.1 in which one of the flips turns up tails and the other two turn up heads.

distinct ways of having two heads and a tail turning up. In general, one computes the number ways in which one can end up with n heads and $N - n$ tails in N flips of a coin by imagining the results of the flip being lined up as in Figure 1.1. Then one shuffles the coins in all possible ways, leading to the factor $N!$, which one divides by the number of ways of shuffling the n heads among themselves and the number of ways of shuffling the $N - n$ tails among themselves (Boas, 1983).

The factor in (1.4) is clearly the one that accounts for all distinct walks. It is not hard to see that the combinatorial factor $N!/n!(N - n)!$ is also equal to the number of different ways that the walker can take n steps to the right and $N - n$ steps to the left. Put another way, the factor $N!/n!(N - n)!$ *is equal to the number of walks that consist of n steps to the right and N − n steps to the left.*

All this leads to the result that the likelihood that the one-dimensional walker will take n steps to the right and $N - n$ steps to the left is

$$\frac{1}{2^N}\frac{N!}{n!(N-n)!} \tag{1.5}$$

Exercise 1.1

How does the result (1.5) change when the coin is "biased" and the probability of a heads at each toss is $p \neq 1/2$? Assume that p does not change from one coin toss to the next.

We can recast our expressions in terms of the location of the walker. Using (1.2) and (1.3), we have for the number of N-step walks that take the walker a distance