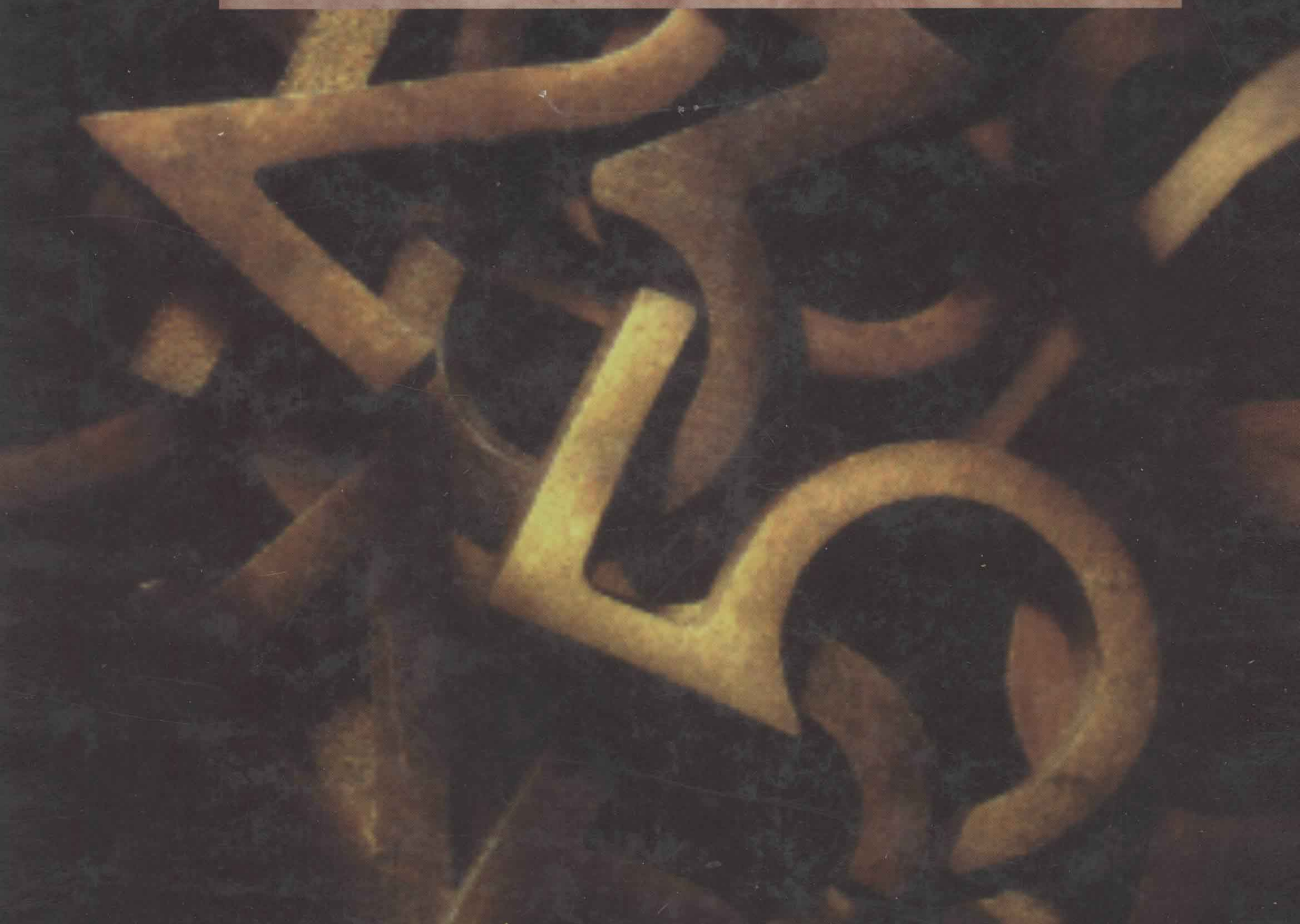
Principles and Practice of

Mathematics



COMAP

Principles and Practice of Mathematics

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Springer

Volume Editors

Kokichi Futatsugi
Japan Advanced Institute of Science and Technology (JAIST)
Ishikawa, Japan
E-mail: kokichi@jaist.ac.jp

Jean-Pierre Jouannaud
École Polytechnique, Laboratoire d'Informatique (LIX)
91128 Palaiseau Cedex, France
E-mail: Jean-Pierre.Jouannaud@lix.polytechnique.fr

José Meseguer
University of Illinois at Urbana-Champaign (UIUC), Computer Science Dept.
Urbana, IL 61801-2302, USA
E-mail: meseguer@cs.uiuc.edu

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Project Director
Solomon Garfunkel, *COMAP, Inc.*

Coordinating Editor
Walter Meyer, *Adelphi University*

AUTHORS

David C. Arney, *USMA, West Point*

Robert Bumcrot, *Hofstra University*

Paul Campbell, *Beloit College*

Joseph Gallian, *University of Minnesota, Duluth*

Frank Giordano, *USMA, West Point*

Rochelle Wilson Meyer, *Nassau County Community College*

Michael Olinick, *Middlebury College*

Alan Tucker, *SUNY at Stony Brook*

CONTRIBUTORS

Sheldon Gordon, *Suffolk County Community College*

Zaven Karian, *Denison University*

PREFACE

Principles and Practice is a revolutionary text which we wrote in order to change the undergraduate mathematics curriculum. Revolutions are started for two reasons: to overturn the status quo, and to realize a vision of the future. Why do we at COMAP think now is the time to incite change through the preparation of this text which, after all, proposes a radically new introductory course for undergraduate mathematics?

The central importance of mathematics in our technologically complex world is undeniable, and the possibilities for new applications are almost endless. But at the undergraduate level, little of this excitement is being conveyed to our students. Currently, attention is being focused on reforming calculus, the traditional gateway course into the undergraduate curriculum. No one is questioning the importance and beauty of continuous mathematics. However, reformed or not, calculus is one branch (and a highly technical one) of a very rich subject. We know the breadth and richness of our subject; how, then, do we expect the students who are starting their study to gain these insights?

In seeking answers to this question, we identified a model which has been in place throughout college science curricula. Every science department offers an introductory course that focuses on developing basic principles and concepts, and at the same time introduces students to the range of the subject—chemistry, physics, biology, etc. The "101–102" sequence usually serves as a prerequisite for further courses. Our proposed new start or gateway into the college mathematics curriculum is only a revolutionary idea for our discipline; other disciplines have had such courses in place for years.

Our project started more than five years ago, with funding from the Division of Undergraduate Education of the National Science Foundation. We asked ourselves a simple question: in designing the first undergraduate course for math and science majors, what should such a course look like? The contents of *Principles and Practice of Mathematics* represents our most considered answer to this provocative question. The course content stresses the breadth of mathematics, discrete and continuous, probabilistic as well as deterministic, algorithmic and conceptual. We emphasize applications that are both real and immediate. And the text includes topics from modern mathematics that are currently homeless in the undergraduate curriculum.

We should stress that the level of mathematics included here is not trivial. The audience for this text should not be confused with that of terminal courses such as surveys of mathematics for liberal arts students, or finite mathematics. Throughout the writing, class-testing, and revising, we have aimed for a level of presentation equivalent to a conventional first-year calculus course. The typical student for this text will have completed the standard prerequisites for studying calculus.

Even for those students who take just a year of college-level mathematics, we will have achieved something of considerable value. For the year's worth of attention and effort they give to mathematics, they will gain a wider understanding of what mathematics is all about, including some of its most modern ideas and applications. Currently, if a student drops out of mathematics after a year of calculus, he or she has no idea of how mathematics provides a conceptual base for computer science, has only a limited concept of abstraction, and, perhaps most damning as we approach the twenty-first century, has seen little or no mathematics more modern than the eighteenth century. Such a student is unaware of subjects such as graph theory, linear programming, and combinatorial optimization, subjects which are taught to beginning students in other disciplines and which appear from time to time in newspapers and popular science magazines.

The adoption of this text, we realize, might mean redesigning the curriculum—and that will not happen overnight. Because of the variety in kinds of schools and programs of study, we expect many trial sections and experimental courses to be offered. While the book is intended for use over two semesters, it is organized so that chapters and sections can be covered selectively and adapted easily for one-term courses.

This book is a team effort in which authors looked over each other's shoulders during numerous team meetings. However, primary writing responsibilities were: David Arney and Frank Giordano, Chapter 1; Robert Bumcrot, Chapter 2; Alan Tucker, Chapter 3; Rochelle Wilson Meyer, Chapters 4, 6, 7;

Paul Campbell, Chapter 5; Michael Olinick, Chapter 8; Joseph Gallian, Chapter 9. The editor wishes to thank each of these authors for being splendid team players as well as talented expositors.

A revolution is the work of many hands, and a project of this magnitude could not be completed without extensive assistance. In particular we would like to thank the National Science Foundation for its steadfast support under the DUE program.

We were fortunate to have a highly talented group of project advisors: Saul Gass, Andrew Gleason, Zaven Karian, Joseph Malkevitch, David Moore, Henry Pollak, Paul Sally, Laurie Snell, Marcia Sward, and Gail Young.

In addition, special thanks go to: John Burns, for careful reading of early drafts, Sheldon Gordon who contributed much to the early work on Chapter 1, Zaven Karian for technology advice, Harald Ness who wrote many of the problems in Chapter 8, and Yves Nievergelt for contributing spotlights.

We also wish to thank all the participants at the West Point workshops of 1994 and 1995, including especially P. Baker, M. Gallit, A. Lebow, C. Lindsey, J. Orlett, B. Reid, P. Rose, S. Seltzer, F. Serio, M. Vanisko; field testers of the early drafts; those who have read and criticized early drafts, including R. Bradley, J. Buonocristiani, M. Fegan, M. Grady, R. Griego, D. Knee, P. Lindstrom, F. Meyer, T. Walsh, W. Williams, J. Wynn.

Last, but not least, we would like to thank our editors, Jerry Lyons, Liesl Gibson, and Teresa Shields for their commitment to this book.

Principles and Practice of Mathematics

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CHAPTER 1

CHANGE

SECTION 1.1 *Introduction*

Change is all around us. Wherever we look, things are changing. We see it in such varied phenomena as

- ➢ the path of a perfectly thrown pass in football or the regal motion of the planets around the sun,
- ➢ the growth of a human being from infancy through old age or the growth of the entire human population on Earth,
- ➢ the temperature change of a cold potato put into a hot oven or the global warming patterns that threaten our very existence,
- ➢ the growth of money deposited in a bank account or the growth pattern of our balance of trade deficit.

All of these changing phenomena can be investigated mathematically. Many of them can be described very effectively by mathematics, and in such cases we can accurately predict what will happen. For instance, the orbits of the planets are ellipses; knowing that precise mathematical relationship allows scientists to calculate the trajectories of man-made satellites so that they reach the desired target. Similarly, the path of the football is a parabola; the quarterback, like the rocket scientist, wants his "satellite" to arrive at the correct spot at the precise instant that the receiver passes through that point.

Population growth patterns tend to be exponential in nature; if current population trends continue unchanged, the population of Mexico will surpass that of the United States somewhere around the year 2051.

In the present chapter, we will develop some extraordinarily powerful, yet simple, mathematical tools that will enable us to study change in a wide variety of areas. To do so, we must consider some basic ideas. First and foremost, the very fact that a quantity *changes* means that the quantity varies with respect to some other quantity. That is, the quantity of interest to us, say position or temperature or population, depends on some other quantity, say time. Consequently, the quantity of interest will always be a *function* of time or some other *independent variable.* For example, if we roll a bowling ball with a forward velocity of 30 ft/sec, after t seconds the distance (d) is approximately expressed as a function of t by $d = 30t$. An example of a function where the independent variable is not time is the circumference of a circle (C), which is a function of the radius (r) expressed by the formula $C = 2\pi r$.

DEFINITIONS

A *function* is a rule or procedure for producing output values from input values of the independent variable.

The set of the possible values of the independent variable is known as the *domain* of the function.

The possible values for the dependent variable are known as the *range* of the function.

Because the independent variable represents an actual quantity, in the real world, such as time, it naturally is limited in terms of the values we can intelligently use. This set of the possible values of the independent variable is known as the *domain* of the function. Similarly, because the quantity of interest, the *dependent variable,* represents an actual quantity, it is also limited in terms of the values it can assume. These possible values for the dependent variable are known as the *range* of the function.

For instance, if we are considering the population of North America over time, then the domain might be limited to the interval from −15,000 (that is, 15,000 B.C., approximately when anthropologists believe the first settlers crossed the land bridge between Siberia and Alaska) to A.D. 2100 (it is extremely difficult to extrapolate very far into the future with any hope of accuracy). The range for population values would then extend from a minimum of zero to a maximum of potentially half a billion (approximately double the present population). Alternatively, if we are interested in the population of the United States, then the domain is limited from 1776 to 2100, say, and the range would be from 2.75 million to about half a billion.

Let's consider how we might represent functions. In previous courses you were probably led to believe that all functions are expressed as an explicit formula of the form $y = f(x)$. While this is true of many situations, we often have to deal with cases where no such formula is known. For example, in a daily lottery, the winning number is a function of the day—in the sense that for each day, there is a definite winning number associated with it—but we have no formula for the lottery to help us get rich.

The study of population presents interesting issues in how we represent functions. Typically, we begin with a table of values, which is a way of presenting a function. For example, the population of the United States in millions from 1780 to 1990 is presented in Table 1. Here the domain is the 22 years that end in 0: 1780, 1790, . . . , 1990.

These population numbers are all approxima-

TABLE 1 The U.S. Population.

Year	*Population (millions)*	*Year*	*Population (millions)*
1780	2.78	1890	62.95
1790	3.93	1900	75.99
1800	5.31	1910	91.97
1810	7.24	1920	105.71
1820	9.64	1930	122.77
1830	12.87	1940	131.67
1840	17.07	1950	150.70
1850	23.19	1960	179.32
1860	31.44	1970	203.30
1870	39.82	1980	226.55
1880	50.16	1990	248.71

tions. No census is completely accurate, and the number for 1780 was not obtained from a systematic census anyhow (the first official U.S. census occurred in 1790).

We might choose to represent such a function via a graph, as shown in Figure 1, which rises from 0 to the population in 1990, about 250 million. Notice that although the graph is not as precise as the table of values appears to be, it quickly gives us a qualitative feel for the trends present in the change we are observing.

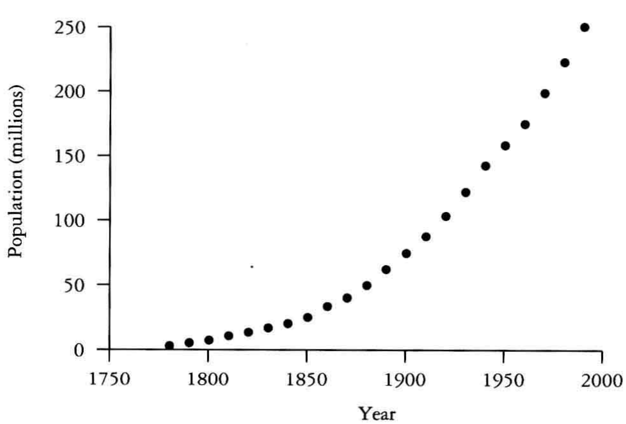


FIGURE 1 U.S. population.

Notice in Figure 1 that the graph is reasonably smooth, showing an upward trend. We will see that we can capture the trend with the following formula (or function), which approximately predicts the U.S. population, $p(x)$, in year x:

$$p(x) = (203{,}211{,}926)2^{0.0216(x-1970)}. \qquad (1)$$

For example, for the year $x = 1980$, the formula predicts

$$\begin{aligned} p(1980) &= (203{,}211{,}926)2^{0.0216(1980-1970)} \\ &= 236{,}032{,}426. \end{aligned}$$

which we can compare to the 226.55 million actually recorded.

We have seen that there are different ways to express a function: as a formula, as a graph, and as a table of observed data. All of these arise in the real world, and we must be able to interpret the behavior of the quantity they represent in each instance.

Equation (1) can clearly have any value of time plugged into it, not just 1780, 1790, 1980, etc. This gives us a way to estimate the population for intermediate times, like 1982, 1943.78, and so on. If we wish to do this, we are saying, in effect, that the domain of the function now consists of all the infinitely many numerical values between 1780 and 1990, i.e., the interval [1780, 1990]. We could even go out on a limb and use the formula to try to predict the population of 2000, by declaring the domain to be [1780, 2000]. If we were to draw the graph of the function with the interval [1780, 2000] as its domain, we get an unbroken curve like that of Figure 2 instead of a series of dots.

We have now seen a number of examples of functions and domains, so let's summarize the important points.

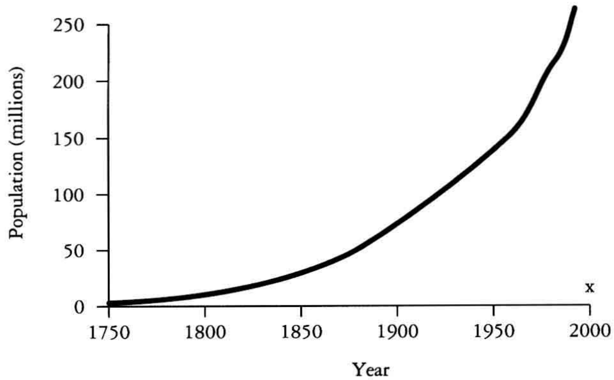


FIGURE 2 A continuous curve for the U.S. population.

TABLE 2 Important Characteristics of a Function.

When is it increasing?
When is it decreasing?
What is its maximum value?
What is its minimum value?
When is the rate of increase increasing?
When is the rate of increase decreasing?
When is the rate of decrease decreasing?
When is the rate of decrease increasing?
When is it increasing or decreasing most rapidly?
What are its roots? (When is its value equal to 0?)
Is it periodic? (Do the function values form a repeating pattern?)

One quantity y, often called the dependent variable, is said to be a function of another quantity x, often called the independent variable, if there is a certain domain of x-values where each x-value has a single, definite y-value associated with it. A set from which these associated y-values come is called the range. The domain and range can be finite or infinite sets. Not all functions can be represented by a formula, although most of the ones of interest to us will be. Tables and graphs are other ways in which functions can be represented.

In particular, when we speak of the behavior of a function or a quantity, there are several specific aspects that are typically important to know, as shown in Table 2.

Difference Equation Models and Their Solutions

In this chapter we will learn about a particular kind of mathematical model called a *difference equation model.* We will learn how to build such a model, find what it means to solve such a model, and see how solutions can be found. The solutions will be functions, represented as formulas, graphs, tables, or sequences of numbers. Finally, we will learn how to analyze our solutions to determine the characteristics outlined in Table 2. Let's preview what is in store for us.

Discrete and Continuous Change

In many cases, the behavior we are observing changes abruptly at instants of time separated by periods when no change occurs (Figure 3). For example, the amount owed on a mortgage or car loan changes when interest is charged and a payment is made. The value of a stock portfolio changes when a dividend is declared or the market value of a share changes at

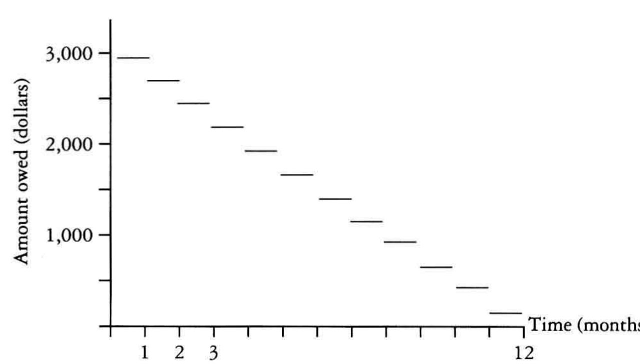
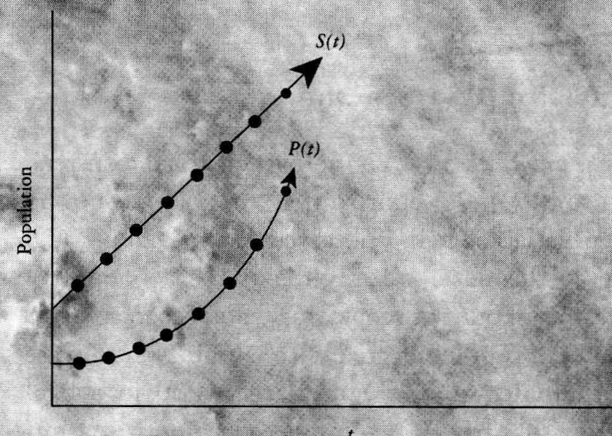


FIGURE 3 Amount of a loan still outstanding at various times.

Population, Misery, and Vice: Is There a Connection?

Sometimes it's fun to be part of a crowd—for example, at this ticker-tape parade on New York's Broadway. But what if there were crowds everywhere, all the time? The most famous western thinker to warn of the perils of too many people was Thomas Malthus, a country parson with a taste for mathematics. Malthus noticed that many people in 18th century England lived amidst misery and vice. He tried to explain this with a mathematical argument that today we can give in the language of difference equations. Malthus claimed that the food supply grows by a constant rate, so the population that can be sustained does also, as shown by the "sustainable population" curve $S(t)$ in the accompanying figure. When not choked off by excessive death rates due to starvation, disease, and warfare, population grows according to a graph like $P(t)$ in the figure. The figure shows a scenario where the population is small in the beginning—many more could be sustained by the available food. However, population grows faster than food supply, and eventually, when $P(t)$ approaches $S(t)$, people can barely get enough to stay alive. At this point, deaths due to what Malthus called misery and vice (starvation, infanticide, murder, etc.) rise to stop the population curve from following its exponentially increasing sweep.



People argue about Malthus' ideas to this day. Do you think there are parts of the world where population pressures are causing hardships? Are there places where the population is not a problem? Is the U.S. birthrate too high or too low?