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Nonlinear Dynamics of Chaotic and Stochastic Systems

Tutorial and Modern Developments

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An ever increasing number of scientific disciplines deal with complex systems. These are systems that are composed of many parts which interact with one another in a more or less complicated manner. One of the most striking features of many such systems is their ability to spontaneously form spatial or temporal structures. A great variety of these structures are found, in both the inanimate and the living world. In the inanimate world of physics and chemistry, examples include the growth of crystals, coherent oscillations of laser light, and the spiral structures formed in fluids and chemical reactions. In biology we encounter the growth of plants and animals (morphogenesis) and the evolution of species. In medicine we observe, for instance, the electromagnetic activity of the brain with its pronounced spatio-temporal structures. Psychology deals with characteristic features of human behavior ranging from simple pattern recognition tasks to complex patterns of social behavior. Examples from sociology include the formation of public opinion and cooperation or competition between social groups.

In recent decades, it has become increasingly evident that all these seemingly quite different kinds of structure formation have a number of important features in common. The task of studying analogies as well as differences between structure formation in these different fields has proved to be an ambitious but highly rewarding endeavor. The Springer Series in Synergetics provides a forum for interdisciplinary research and discussions on this fascinating new scientific challenge. It deals with both experimental and theoretical aspects. The scientific community and the interested layman are becoming ever more conscious of concepts such as self-organization, instabilities, deterministic chaos, nonlinearity, dynamical systems, stochastic processes, and complexity. All of these concepts are facets of a field that tackles complex systems, namely synergetics. Students, research workers, university teachers, and interested laymen can find the details and latest developments in the Springer Series in Synergetics, which publishes textbooks, monographs and, occasionally, proceedings. As witnessed by the previously published volumes, this series has always been at the forefront of modern research in the above mentioned fields. It includes textbooks on all aspects of this rapidly growing field, books which provide a sound basis for the study of complex systems.

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To our teachers and friends:

Werner Ebeling, Yuri L. Klimontovich
and Frank Moss

Preface

This book is devoted to the classical background and to contemporary results on nonlinear dynamics of deterministic and stochastic systems. Considerable attention is given to the effects of noise on various regimes of dynamic systems with noise-induced order.

On the one hand, there exists a rich literature of excellent books on nonlinear dynamics and chaos; on the other hand, there are many marvelous monographs and textbooks on the statistical physics of far-from-equilibrium and stochastic processes. This book is an attempt to combine the approach of nonlinear dynamics based on the deterministic evolution equations with the approach of statistical physics based on stochastic or kinetic equations. One of our main aims is to show the important role of noise in the organization and properties of dynamic regimes of nonlinear dissipative systems.

We cover a limited region in the interesting and still expanding field of nonlinear dynamics. Nowadays the variety of topics with regard to deterministic and stochastic dynamic systems is extremely large. Three main criteria were followed in writing the book and to give a reasonable and closed presentation: (i) the dynamic model should be minimal, that is, most transparent in the physical and mathematical sense, (ii) the model should be the simplest which nevertheless clearly demonstrates most important features of the phenomenon under consideration, and (iii) most attention is paid to models and phenomena on which the authors have gained great experience in recent years.

The book consists of three chapters. The first chapter serves as a brief introduction, giving the fundamental background of the theory of nonlinear deterministic and stochastic systems and a classical theory of the synchronization of periodic oscillations. All basic definitions and notions necessary for studying the subsequent chapters without referring to special literature are presented.

The second chapter is devoted to deterministic chaos. We discuss various scenarios of chaos onset, including the problem of the destruction of two- and three-frequency quasiperiodic motion. Different aspects of synchronization and chaos control as well as the methods of reconstruction of attractors and dynamic systems from experimental time series are also discussed.

The third chapter is concerned with stochastic systems whose dynamics essentially depend on the influence of noise. Several nonlinear phenomena are discussed: stochastic resonance in dynamic systems subjected to harmonic and complex signals and noise, stochastic synchronization and stochastic ratchets, which are the noise-induced ordered and directed transport of Brownian particles moving in bistable and periodic potentials. Special attention is given to the role of noise in excitable dynamics.

The book is directed to a large circle of possible readers in the natural sciences. The first chapter will be helpful for undergraduate and graduate students in physics, chemistry, biology and economics, as well as for lecturers of these fields interested in modern problems of nonlinear dynamics. Specialists of nonlinear dynamics may use this part as an extended dictionary. The second and the third chapters of the book are addressed to specialists in the field of mathematical modeling of the complex dynamics of nonlinear systems in the presence of noise.

We tried to write this book in such a manner that each of the three chapters can be understood in most parts independently of the others. Particularly, each chapter has its own list of references. This choice is based on the desire to be helpful to the reader. Undoubtedly, the lists of references are incomplete, since there exists an enormously large number of publications which are devoted to the topics considered in this book.

This book is a result of the long-term collaboration of the Nonlinear Dynamics Laboratory at Saratov State University, the group of Applied Stochastic Processes of Humboldt University at Berlin, and the Center for Neurodynamics at the University of Missouri at St. Louis. We want to express our deep gratitude to W. Ebeling, Yu.L. Klimontovich and F. Moss for their support, scientific exchange and constant interest. We acknowledge fruitful discussions with C. van den Broeck, P. Hänggi, J. Kurths, A. Longtin, A. Pikovski and Yu.M. Romanovski. The book has benefited a lot from our coauthors of the original literature. We wish to thank A. Balanov, R. Bartussek, V. Bucholtz, I. Diksthein, J.A. Freund, J. García-Ojalvo, M. Hasler, N. Janson, T. Kapitaniak, I. Khovanov, M. Kostur, P.S. Landa, B. Lindner, P. McClintock, E. Mosekilde, A. Pavlov, T. Pöschel, D. Postnov, P. Reimann, R. Rozenfeld, P. Ruszczynsky, A. Shabunin, B. Shulgin, U. Siewert, A. Silchenko, O. Sosnovtseva, A. Zaikin and C. Zülicke for regular and fruitful discussions, criticism and valuable remarks which give us deeper insight into the problems we study.

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Saratov, Berlin and St. Louis
December 2001

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1. Tutorial

1.1 Dynamical Systems

1.1.1 Introduction

The knowledge of nonlinear dynamics is based on the notion of a *dynamical system* (DS). A DS may be thought of as an object of any nature, whose state evolves in time according to some dynamical law, i.e., as a result of the action of a *deterministic* evolution operator. Thus, the notion of DS is the result of a certain amount of idealization when random factors inevitably present in any real system are neglected.

The theory of DS is a wide and independent field of scientific research. The present section addresses only those parts, which are used in the subsequent chapters of this book. The main attention is paid to a linear analysis of the stability of solutions of ordinary differential equations. We also describe local and nonlocal bifurcations of typical limit sets and present a classification of attractors of DS.

The structure of chaotic attractors defines the properties of regimes of deterministic chaos in DS. It is known that the classical knowledge of dynamical chaos is based on the properties of robust hyperbolic (strange) attractors. Besides hyperbolic attractors, we also consider in more detail the peculiarities of nonhyperbolic attractors (quasiattractors). This sort of chaotic attractor reflects to a great extent the properties of deterministic chaos in real systems and serves as the mathematical image of experimentally observed chaos.

1.1.2 The Dynamical System and Its Mathematical Model

A DS has an associated *mathematical model.* The latter is considered to be defined if the *system state* as a set of some quantities or functions is determined and an *evolution operator* is specified which gives a correspondence between the initial state of the system and a unique state at each subsequent time moment. The evolution operator may be represented, for example, as a set of differential, integral and integro-differential equations, of discrete maps, or in the form of matrices, graphs, etc. The form of the mathematical model of the DS under study depends on which method of description is chosen.

Depending on the approximation degree and on the problem to be studied, the same real system can be associated with principally different mathematical models, e.g., a pendulum with and without friction. Moreover, from a qualitative viewpoint, we can often introduce into consideration a DS, e.g., the cardio-vascular system of a living organism, but it is not always possible to define its mathematical model.

DS are classified based on the form of state definition, on the properties and the method of description of the evolution operator. The set of some quantities x_j, $j = 1, 2, \ldots, N$, or functions $x_j(\boldsymbol{r})$, $\boldsymbol{r} \in \mathbf{R}^M$ determines the state of a system. Here, x_j are referred to as *dynamical variables*, which are directly related to the quantitative characteristics observed and measured in real systems (current, voltage, velocity, temperature, concentration, population size, etc.). The set of all possible states of the system is called its *phase space*. If x_j are variables and not functions and their number N is finite, the system phase space $\mathbf{R}^N$ has a finite dimension. Systems with finite-dimensional phase space are referred to as those with *lumped parameters*, because their parameters are not functions of spatial coordinates. Such systems are described by ordinary differential equations or return maps.

However, there is a wide class of systems with infinite-dimensional phase space. If the dynamical variables x_j of a system are functions of some variables r_k, $k = 1, 2, \ldots, M$, the system phase space is infinite-dimensional. As a rule, r_k represent spatial coordinates, and thus the system parameters depend on a point in space. Such systems are called *distributed parameter* or simply *distributed* systems. They are often represented by partial differential equations or integral equations. One more example of systems with infinite-dimensional phase space is a system whose evolution operator includes a time delay, T_d. In this case the system state is also defined by the set of functions $x_j(t)$, $t \in [0, T_\mathrm{d}]$.

Several classes of DS can be distinguished depending on the properties of the evolution operator. If the evolution operator obeys the property of superposition, i.e., it is linear and the corresponding system is *linear*; otherwise the system is *nonlinear*. If the system state and the evolution operator are specified for any time moment, we deal with a *time-continuous* system. If the system state is defined only at separate (discrete) time moments, we have a system with *discrete time* (*map* or *cascade*). For cascades, the evolution operator is usually defined by the *first return function*, or *return map*. If the evolution operator depends implicitly on time, the corresponding system is *autonomous*, i.e., it contains no additive or multiplicative external forces depending explicitly on time; otherwise we deal with a *nonautonomous* system. Two kinds of DS are distinguished, namely, *conservative* and *nonconservative*. For a conservative system, the volume in phase space is conserved during time evolves. For a nonconservative system, the volume is usually contracted. The contraction of phase volume in mechanical systems corresponds to lost of energy as result of dissipation. A growth of phase volume implies a supply of energy to the system which can be named negative dissipation. Therefore, DS in which the energy or phase volume varies are called *dissipative systems.*

Among a wide class of DS, a special place is occupied by systems which can demonstrate oscillations, i.e., processes showing full or partial repetition. Oscillatory systems, as well as DS in general, are divided into *linear* and *nonlinear*, *lumped* and *distributed*, *conservative* and *dissipative*, and *autonomous* and *nonautonomous.* A special class includes the so-called *self-sustained systems.*

Nonlinear dissipative systems in which nondecaying oscillations can appear and be sustained without any external force are called *self-sustained*, and oscillations themselves in such systems are called *self-sustained oscillations.* The energy lost as dissipation in a self-sustained system is compensated from an external source. A peculiarity of self-sustained oscillations is that their characteristics (amplitude, frequency, waveform, etc.) do not depend on the properties of a power source and hold under variation, at least small, of initial conditions [1].

Phase Portraits of Dynamical Systems. A method for analyzing oscillations of DS by means of their graphical representation in phase space was introduced to the theory of oscillations by L.I. Mandelstam and A.A. Andronov [1]. Since then, this method has become the customary tool for studying various oscillatory phenomena. When oscillations of complex form, i.e., dynamical chaos, were discovered, this method increased in importance. The analysis of phase portraits of complex oscillatory processes allows one to judge the topological structure of a chaotic limit set and to make sometimes valid guesses and assumptions which appear to be valuable when performing further investigations [2].

Let the DS under study be described by ordinary differential equations

$$\dot{x_j} = f_j(x_1, x_2, \ldots, x_N)\,, \tag{1.1}$$

where $j = 1, 2, \ldots, N$, or in vector form

$$\dot{\boldsymbol{x}} = \boldsymbol{F}(\boldsymbol{x})\,. \tag{1.2}$$

$\boldsymbol{x}$ represents a vector with components x_j, the index j runs over $j = 1, 2, \ldots, N$, and $\boldsymbol{F}(\boldsymbol{x})$ is a vector-function with components $f_j(\boldsymbol{x})$. The set of N dynamical variables x_j or the N-dimensional vector $\boldsymbol{x}$ determines the system state which can be viewed as a point in state space $\mathbf{R}^N$. This point is called a *representative* or *phase* point, and the state space $\mathbf{R}^N$ is called the *phase space* of DS. The motion of a phase point corresponds to the time evolution of a state of the system. The trajectory of a phase point, starting from some initial point $\boldsymbol{x}_0 = \boldsymbol{x}(t_0)$ and followed as $t \to \pm\infty$, represents a *phase trajectory.* A similar notion of integral curves is sometimes used. These curves are described by equations $\mathrm{d}x_j/\mathrm{d}x_k = \Phi(x_1, x_2, \ldots, x_N)$, where x_k is one of the dynamical variables. An integral curve and a phase trajectory often coincide, but the integral curve may consist of several phase trajectories if it passes through a singular point. The right-hand side of (1.2) defines the velocity vector field $\boldsymbol{F}(\boldsymbol{x})$ of a phase point in the system phase space. Points in phase space for which $f_j(\boldsymbol{x}) = 0$, $j = 1, 2, \dots, N$, remain unchanged with time. They are called *fixed points*, *singular points* or *equilibrium points* of the DS. A set of characteristic phase trajectories in the phase space represents the *phase portrait* of the DS.

Besides the phase space dimension N, the *number of degrees of freedom* $n = N/2$ is often introduced. This tradition came from mechanics, where a system is considered as a set of mass points, each being described by a second-order equation of motion. n generalized coordinates and n generalized impulses are introduced so that the total number of dynamical variables $N = 2n$ appears to be even and the number of independent generalized coordinates n (the number of freedom degrees) an integer. For an arbitrary DS (1.1) the number of degrees of freedom will be, in general, a multiple of 0.5.

Consider the harmonic oscillator

$$\ddot{x} + \omega_0^2 x = 0\,. \tag{1.3}$$

Its phase portrait is shown in Fig. 1.1a and represents a family of concentric ellipses (in the case $\omega_0 = 1$, circles) in the plane $x_1 = x$, $x_2 = \dot{x}$, centered at the origin of coordinates:

$$\frac{\omega_0^2 x_1^2}{2} + \frac{x_2^2}{2} = H(x_1, x_2) = \text{const.} \tag{1.4}$$

Each value of the total energy $H(x_1, x_2)$ corresponds to its own ellipse. At the origin we have the equilibrium state called a *center*. When dissipation is added to the linear oscillator, phase trajectories starting from any point in the phase plane approach equilibrium at the origin in the limit as $t \to \infty$. The phase trajectories look like spirals twisting towards the origin (Fig. 1.1b) if dissipation is low and the solutions of the damped harmonic oscillator correspond to decaying oscillations. In this case the equilibrium state is a *stable focus*. With an increasing damping coefficient, the solutions become aperiodic and correspond to the phase portrait shown in Fig. 1.1c with the equilibrium called a *stable node*.

By using a potential function $U(x)$, it is easy to construct qualitatively the phase portrait for a nonlinear conservative oscillator which is governed by

$$\ddot{x} + \frac{\mathrm{d}U(x)}{\mathrm{d}x} = 0\,.$$

An example of such a construction is given in Fig. 1.2. Minima of the potential function conform to the center-type equilibrium states. In a potential well about each center, a family of closed curves is arranged which correspond to different values of the integral of energy $H(x, \dot{x})$. In the nearest neighborhood

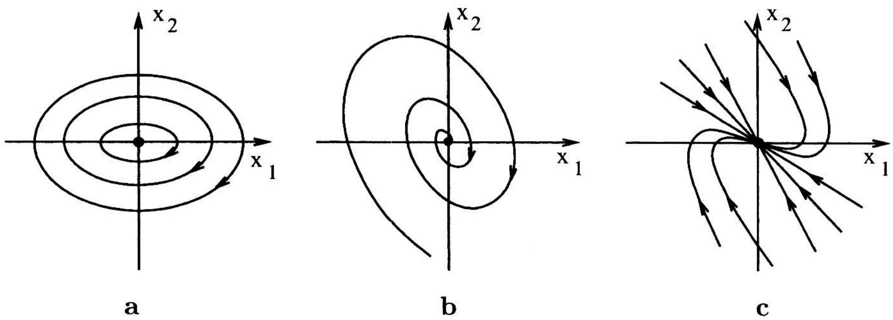

Fig. 1.1. Phase portraits of linear oscillators: (**a**) without dissipation, (**b**) with low dissipation, and (**c**) with high dissipation

of the center these curves have an ellipse-like shape but they are deformed when moving away from the center. Maxima of $U(x)$ correspond to equilibria called *saddles.* Such equilibrium states are unstable. Phase trajectories tending to the saddle Q (Fig. 1.2) as $t \to \pm\infty$ belong to singular integral curves called *separatrices of saddle* Q. A pair of trajectories approaching the saddle forwardly in time forms its *stable maniflod* W_Q^s, and a pair of trajectories

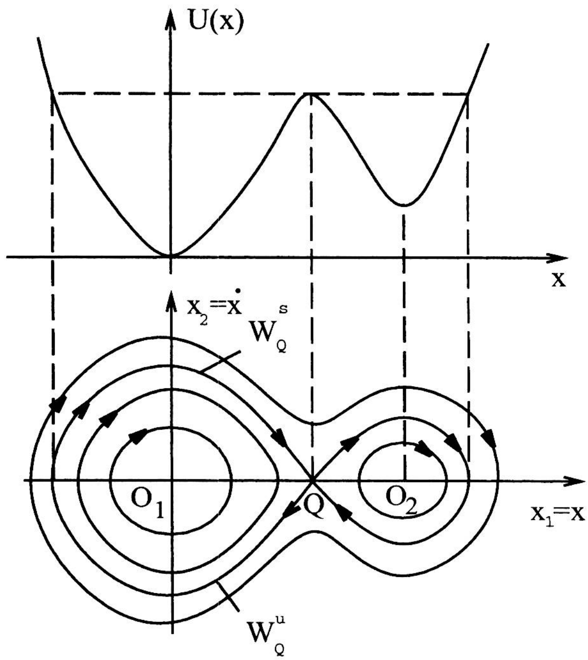

Fig. 1.2. Qualitative construction of the phase portrait of a nonlinear conservative oscillator using the potential function $U(x)$

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