Yakov G. Sinai

Selecta

VOLUME I

Ergodic Theory and Dynamical Systems

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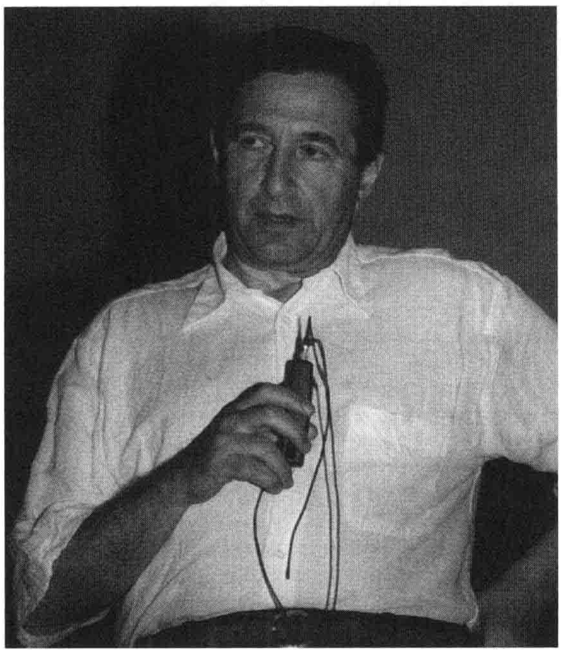

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Selecta

Preface

Two volumes of Selecta contain many of Yakov Sinai's papers spanning more than half of a century. Some of these papers became classics a long time ago, some later, and others were published only recently.

Sinai has pioneered building bridges between the theory of dynamical systems and statistical mechanics. Often switching his main research interests from one field to another, he has demonstrated how the ideas and approaches in one area can enrich and bring new understanding to the other one. This Selecta clearly demonstrates how Sinai succeeded in transforming these two areas essentially into one with a unified vision and a wealth of tools and ideas. This singular vision can be traced in throughout these papers, including those on mathematical physics, fluid dynamics, and Partial Differential Equations.

Sinai is universally considered as the major architect of the modern theory of dynamical systems. Naturally, the first volume is dedicated to ergodic theory and dynamical systems. The part on entropy theory demonstrates the dramatic beginning of one of the most revolutionary discoveries in mathematics of the twentieth century (which allowed to build a unified theory of probabilistic and deterministic systems). The last paper in this part was published 40 years ago. Chaotic billiards is another flourishing area whose foundations were laid in a pioneering work of Sinai.

The first three papers in the "Dynamical Systems" section have dramatically changed this field by bringing together the concepts and approaches from statistical mechanics and dynamics. These ideas are at the heart of thermodynamic formalism, which has since become the basic approach to the study of strongly chaotic (hyperbolic) systems. Other papers in this volume include an influential paper on Feigenbaum universality and more recent papers related to number-theoretic problems.

The selection of the papers for this edition was made by Sinai himself. He has also provided commentary for each paper. (It was a tough selection, and in our opinion quite a few of Sinai's classical papers were not included here.) The reader of this Selecta will be impressed by the variety of brilliant and unconventional ideas which revolutionized so many areas of mathematics and created so many exciting new directions. Sinai's enthusiasm and infinite optimism, multiplied by brilliance and consistent hard work, are responsible for that. Very often his intuition, taste, and enthusiasm have led to a goal visible at the time only by him. Sinai created new mathematical machineries, which were later refined and polished by others while he was busy with other discoveries.

This collection of papers of one of the giants of modern mathematics will serve as an inspiration for students, as well as for senior researchers, demonstrating that an exciting scientific journey through the various disciplines never ends if one is truly fascinated by mathematics. A piece of advice to the readers: Try to borrow some of Sinai's optimism while reading this Selecta and keep that optimism with you in your research.

Leonid A. Bunimovich
Dmitry Dolgopyat
Konstantin M. Khanin

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Ya. G. Sinai

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Curriculum Vitae

Ya. G. Sinai

Birthdate:	September 21, 1935
Birthplace:	Moscow, USSR

Educational History

Doctor Degree	Moscow State University, 1963
Ph.D.	Moscow State University, 1960
B.S.	Moscow State University, 1957

Professional History

1993–present	Professor, Department of Mathematics, Princeton University
1971–1993	Professor, Moscow State University (part of position)
1971–present	Senior Researcher, Landau Institute of Theoretical Physics, Academy of Sciences of Russia
1960–1971	Scientific Researcher, Laboratory of Probabilistic & Statistical Methods, Moscow State University

Lectures

1962	Invited Speaker, International Congress of Mathematics, Stockholm
1970	Invited Speaker, International Congress of Mathematics, Nice
1978	Loeb Lecturer, Harvard University
1981–1986	Plenary Speaker, International Congresses on Mathematical Physics, Berlin (1981), Marseilles (1986)
1989	Distinguished Lecturer, Israel
1990	S. Lefshetz Lectures, Mexico
1990	Plenary Speaker, International Congress of Mathematics, Kyoto
1993	Landau Lectures, Hebrew University of Jerusalem

Awards & Honors

1983	Foreign Honorary Member, American Academy of Arts & Science
1986	Boltzman Gold Medal given by IUPAP
1989	Heineman Prize given by the American Physical Society
1990	Markov Prize given by the Russian Academy of Sciences
1991	Member of Russian Academy of Sciences
1992	Honorary Member, London Mathematical Society
1992	Dirac Medal given by ICPT in Trieste
1993	Doctor Honoris Causa of Warsaw University
1993	Foreign Member of Hungarian Academy of Sciences
1997	Wolf Prize in Mathematics
1999	Foreign Associate of the National Academy of U.S.A.
2000	Foreign Member of the Brazilean Academy of Sciences
2000	Brazilean Award of Merits in Sciences
2001	Moser Prize given by SIAM
2002	Nemmers Prize in Mathematics
2002	Honorary degree from Budapest University of Science and Technology

2005	Honorary degree from Hebrew University in Jerusalem
2008	Lagrange Prize, given by ISI Institute, Torino, Italy
2008	Member of the Academia Europaea
2009	Foreign Member of the Royal Society of London
2009	Henri Poincaré Prize given by the International Association of Mathematical Physics
2009	Dobrushin International Prize given by the Institute of Information Transmission, Russian Academy of Sciences
2009	Polish Academy of Sciences, Foreign Member

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Part III. The Theory of Hyperbolic Dynamical Systems: Markov Partitions and Thermodynamic Formalism

Part IV. Billiards

Part I
Entropy Theory of Dynamical Systems

On the Notion of Entropy of a Dynamical System*

Ya. G. Sinai

§1. In this paper we define the entropy of an arbitrary automorphism of a Lebesgue space. In certain cases it can be calculated using Theorem 1 that follows the definition. Thus we get new metric invariants of some ergodic automorphisms of compact abelian groups.

§2. Let M be a Lebesgue space, with σ-algebra S of measurable subsets and probability measure μ, and let T be an arbitrary automorphism of this space (see [2]). By a finite partition A of M we mean a finite collection of pair-wise disjoint subsets[1] $A_1, \dots, A_n$ whose union is M. Then $T^k A$ is the partition generated by $\{T^k A_i\}, i = 1, \dots, n$.
The entropy of a finite partition A is given by the well-known formula

$$h(A) = -\sum_{i=1}^{n} \mu(A_i) \log \mu(A_i).$$

Using this formula one can find the value of the entropy of the partition $A \vee TA \vee \ldots \vee T^n A$ whose elements are $A_{i_0} \cap TA_{i_1} \cap \ldots \cap T^n A_{i_n}$. From general theorems of information theory (see [4]) it follows that the limit

$$\lim_{k \to \infty} \frac{h(A \vee TA \vee \ldots \vee T^k A)}{k+1} = h_T(A)$$

exists for any finite partition A.

Definition. The entropy of an automorphism T is the supremum of $h_T(A)$ over all finite partitions A: $h_T = \sup_A h_T(A)$.

Consider two finite partitions $A = \{A_1, \dots, A_n\}$ and $B = \{B_1, \dots, B_l\}$ and suppose that all subsets B_i belong to the closed σ-algebra generated by subsets $\{T^n A_j,\ 1 \leqslant j \leqslant k,\ -\infty < n < \infty\}$. Then we have the following theorem.

*Free translation of the Author. Original article: On the notion of Entropy of a Dynamical System (in Russian), *Doklady of the Soviet Academy of Sciences of USSR*, 1959, v. **124**, N.4, 768-771.

[1] All sets are measurable.

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Theorem 1. *For partitions A, B as above we have $h_T(B) \leqslant h_T(A)$.*

Let us formulate general properties of entropy which we shall use below. Let K, L, M be arbitrary finite partitions. Then

(1) $h(K \vee L) = h(K) + h(L|K)$,

(2α) $h(K) \leqslant h(K \vee L) \leqslant h(K) + h(L)$,

(2β) $h(K \vee L|M) \leqslant h(K|M) + h(L|M)$,

(3) $h(K|L) \geqslant h(K|M)$,

if elements of L are unions of elements of M.

Proof. From (1) and (2α)

$$\begin{aligned} &h(B \vee TB \vee \ldots \vee T^r B) \\ &\quad \leqslant h(B \vee TB \vee \ldots \vee T^r B \vee T^{-n} A \vee \ldots \vee T^{n+r} A) \\ &\quad = h(T^{-n} A \vee \ldots \vee T^{n+r} A) + h(B \vee \ldots \vee T^r B | T^{-n} A \vee \ldots \vee T^{n+r} A). \quad (1) \end{aligned}$$

From (2β) and (3)

$$\begin{aligned} &h(B \vee TB \vee \ldots \vee T^r B | T^{-n} A \vee \ldots \vee T^{n+r} A) \\ &\quad \leqslant \sum_{i=0}^{r} h(T^i B | T^{-n} A \vee \ldots \vee T^{n+r} A) \leqslant \sum_{i=0}^{r} h(T^i B | T^{-n+i} A \vee \ldots \vee T^{n+i} A) \\ &\quad = (r+1) h(B | T^{-n} A \vee \ldots \vee T^n A). \quad (2) \end{aligned}$$

It is easy to show that our condition on partitions A and B implies that for every $\varepsilon > 0$ the conditional entropy $h(B|T^{-n}A \vee \ldots \vee T^n A) < \varepsilon$ if n is sufficiently large. Dividing both sides of (1) by r and using (2) and the last statement we get

$$\frac{h(B \vee TB \vee \ldots \vee T^r B)}{r+1} \leqslant \frac{h(A \vee \ldots \vee T^{2n+r} A)}{2n+r+1} \cdot \frac{2n+r+1}{r} + \varepsilon.$$

Since n depends only on ε we can let $r \to \infty$ and get the result because ε is arbitrary small. □

Corollary 1. *If a partition A is such that the closed σ-algebra generated by all sets $\{T^k A_i\}, -\infty < k < \infty, 1 \leqslant i \leqslant n$ is* S *then $h_T = h_T(A)$.*

Theorem 2. *If a partition A is such the closed σ-algebra generated by $\{T^k A_i\}$, $0 \leqslant k < \infty, 1 \leqslant i \leqslant n$, is* S *then $h_T = 0$.*

The proof is based on the fact that the condition of the theorem implies that $h(A|TA \vee \ldots \vee T^n A) \to 0$ as $n \to \infty$.