M. MAQUSI

APPLIED WALSH ANALYSIS

HEYDEN

APPLIED WALSH ANALYSIS

MOHAMMAD MAQUSI

Electrical Engineering Department
University of Jordan
Amman, Jordan

HEYDEN

LONDON • PHILADELPHIA • RHEINE

Heyden & Son Ltd, Spectrum House, Hillview Gardens, London NW4 2JQ, UK
Heyden & Son Inc., 247 South 41st Street, Philadelphia, PA 19104, USA
Heyden & Son GmbH, Devesburgstrasse 6, 4440 Rheine, West Germany

British Library Cataloguing in Publication Data

Maqusi, Mohammad
Applied Walsh analysis.
1. Engineering mathematics 2. Walsh functions
I. Title
620'.001'51555 TA347.W/ 80-49885

ISBN 0-85501-162-9

Printed in Great Britain by Maclehose and Co. Ltd, Renfrew and London

To Hiyam, Suhair, and Samar

PREFACE

The material in this book is intended to provide concise and clear treatment of a general working theory of Walsh analysis and some of its applications. In this respect, emphasis is placed on generalized 'continuous' Walsh functions. However, due to the significance of the discrete aspect of Walsh analysis, certain considerations in this latter aspect are also developed throughout the various chapters.

The book is arranged so that it takes the reader from somewhat basic topics on theory and applications to further topics of intermediate and advanced levels. Hence, the material attempts to produce some main techniques in applied Walsh analysis, and advance from somewhat simpler to more sophisticated methods and problems.

The first three chapters provide an introductory working theory of Walsh functions and transforms. Chapter 1 presents some historical remarks on Walsh and Walsh-related orthogonal functions, and reviews the mission of this work. Chapters 2 and 3 are concerned with the introduction of Walsh functions and the Walsh transform. These two chapters establish basic principles in the theory of Walsh analysis.

Chapter 4 introduces some concepts which are fundamental to statistical studies of random processes by Walsh function techniques. Concepts such as dyadic and stayadic correlation, sequency power spectral representation, and a sampling theory are formulated to suit Walsh function analysis.

Chapter 5 deals with a special class of linear systems, called dyadic invariant. The characterization of such systems is made in both time and sequency domains. Furthermore, certain statistical concepts developed in Chapter 4 are utilized in the statistical description of such systems.

Chapter 6, on the other hand, turns to the study of nonlinear systems. Walsh functions are employed in the derivation of useful approximate output expressions for such systems. In another aspect of study, the processing of random signals by nonlinear systems is investigated with the aim of producing convenient expressions for the computation of certain output quantities, such as dyadic and stayadic correlation functions. In addition, some system identification procedures are developed through the use of Walsh input test signals.

Chapter 7 deals with some statistical studies in the applications of Walsh functions. In particular, Walsh series expansions are used for the representation of suitable probability distributions. This in turn facilitates the computations of general moments of nonlinear transformations (systems).

Chapter 8 is not on Walsh functions, but it does introduce a related set of functions which exhibit similar characteristics to Walsh functions. This chapter deals with Haar functions which have often been linked to Walsh functions in their studies and applications.

In Chapter 9 some relations are derived between Walsh and Fourier series and transforms. These relations establish channels of spectral conversion between the sequency and frequency domains. In addition, some conversion relations between the Z-transform and the discrete Walsh transform are also developed.

Chapter 10 deals with certain applications of Walsh analysis in communications. A scheme of sequency-division multiplexing based on Walsh carriers is discussed in comparison with conventional time- and frequency-division multiplexing methods. The data coding problem demonstrates the possible use of Walsh coefficients for signal transmission. And image transmission based on Walsh functions demonstrates certain advantages, such as simplicity of representation and bandwidth compression. The general Walsh signal processing problem is illustrated via considerations in electrocardiograph and speech signal analyses. These latter problems further demonstrate certain advantages which may be gained by employing Walsh transforms in the digital processing of signals.

The appendices contain results which are considered to be supplementary to certain developments contained in the respective chapters. In addition, they provide useful concise references on discussed material.

A large number of selected references appears at the end of each chapter. In addition, a separate list of other selected references provides a supplement to the chapter references. The interested reader should find these references helpful in further study and research on Walsh functions and their applications.

In general, the book is intended to provide a useful reference work especially for the diligent and newly-initiated student of Walsh theory. It is further hoped that it will help interested engineers and applied scientists to gain a good appreciation of Walsh analysis techniques and their applications.

The author wishes to thank L. C. Ludeman of New Mexico State University for his help in checking part of this work, and the proofreading. My deep thanks go to my wife Hiyam and my daughters Suhair and Samar for their endurance during the preparation of this work.

June 1980 M. Maqusi

ABBREVIATIONS AND SYMBOLS

BDF	Bandpass filter
DFT	Discrete Fourier transform
DWT	Discrete Walsh transform
ECG	Electrocardiogram
FDM	Frequency-division multiplexing
FFT	Fast Fourier transform
FHT	Fast Hadamard transform
FWT	Fast Walsh transform
GSBPF	Generalized sequency bandpass filter
GSBSF	Generalized sequency bandstop filter
GSHPF	Generalized sequency highpass filter
GSLPF	Generalized sequency lowpass filter
HT	Hadamard transform
KLT	Karhunen-Loéve transform
LDI	Linear dyadic invariant
LPF	Lowpass filter
LTI	Linear time invariant
PAM	Pulse amplitude modulation
PCM	Pulse code modulation
PDF	Probability density function
SBPF	Sequency bandpass filter
SDM	Sequency-division multiplexing
SLPF	Sequency lowpass filter
TDM	Time-division multiplexing
WCF	Walsh characteristic function
WDS	Weakly dyadic stationary
WSS	Weakly (wide) sense stationary
$C_x(\tau)$	Stayadic correlation function
$D_x(\tau)$	Dayadic correlation function
$D_n(x)$	Walsh-Dirichlet kernel

$\delta(x)$	Delta function, or impulse
δ_{nm}	Kronecker delta
$f(x)$, $f_X(x)$	Probability density function
$F(x)$, $F_X(x)$	Probability distribution function
H	Hadamard matrix
$\mathcal{H}$	Haar matrix
$H(\sigma)$	Sequency transfer function
$J(\sigma,x)$	Fine integral
$R_x(\tau)$	Ordinary correlation function
$S_x(\sigma)$	Walsh-characteristic function
W	Walsh matrix
$wal(n,x)$	Walsh function (sequency-ordered)
$\psi(n,x)$	Walsh–Fine function
$\phi(n,x)$	Rademacher function
$\phi_n^m(x)$	Haar function
$\Gamma_x(\sigma)$	Sequency power spectral density
$\oplus$	Dyadic addition operator
$\circledast$	Dyadic convolution operator
$\frac{d}{dx}^{\oplus}$	Dyadic derivative operator
ϵ	Belong to

CONTENTS

CHAPTER 1 · INTRODUCTION

1. General Remarks

Ever since the French mathematician Joseph B. Fourier offered a solution to a heat conduction problem by a trigonometric series representation [1], Fourier theory has been of major importance in applications to science and engineering problems. In particular, Fourier analysis is well established in the engineering sciences as a means for analog wave analysis where the sine-cosine system of functions finds ample opportunities for use. However, with the advent of digital computers and the introduction of their use in various fields, the theory of discrete (trigonometric) Fourier analysis had to be developed further. In this respect, it is found that other theories can offer equal and sometimes better means of analysis. One such theory is based on a set of functions due to Walsh [2].

Theorists of Walsh functions, especially in the last two decades or so, have been attempting to extend Walsh analysis in a parallel manner to the long-established Fourier analysis. Such attempts have actually resulted in the formulation of basic notions and concepts of Walsh theory. Meanwhile, other efforts have been concentrated on the applications side of Walsh analysis. In this latter region, much progress has also been achieved. However, a next stage of development will most probably be directed towards the applications of Walsh theory, as an independent entity, for the analysis of suitable signals and systems.

In particular, harmonic analysis has been used fruitfully in many aspects of communication theory, especially in the study of time series analysis [3]. Due to the popularity and mathematical tractability of Fourier theory, trigonometric functions have obviously surpassed other sets of complete orthonormal functions. The frequency spectral representation of stationary stochastic processes has been both prominent and useful in studies of such processes [4]. Admittedly, Fourier analysis is well suited for considerations of stationary random processes and linear time invariant systems. However, for a new type of processes called dyadic stationary processes, Walsh analysis is more suitable. On the systems side, sequency spectral representation plays just as an important role in the analysis of linear dyadic invariant systems as do methods of (frequency) spectral analysis for linear time invariant systems. Specifically, for digital-type signals, Walsh function expansions may be applied in facilitating many aspects of computations associated with such signals [5].

2. Orthogonal Functions: Historical Remarks

At the turn of the twentieth century, scientists and engineers were well aware of the existence and use of many orthogonal sets of functions and polynomials [6]. Such sets include trigonometric functions, Bessel functions, Hermite polynomials, Legendre polynomials and a host of others. But as remarked earlier, trigonometric functions which are inherently encountered in Fourier analysis have played a dominant role in applications to engineering and science problems.

The selection and use of any orthogonal set of functions still hinges primarily on the type of problem under study and the consequent amenability of the problem to the specified set. For instance, while some sets render rather simple and useful solutions to a certain problem, other sets give complicated and less useful forms of solution. For example, the discrete Karhunen-Loéve (KL) expansion technique offers an efficient representation for discrete time random process. But the technique is usually very difficult to manipulate and implement. On the other hand, for such a process, Walsh function analysis may prove more advantageous from such points of view as manipulation and implementation.

By the start of the twentieth century, some mathematicians sought the construction of orthogonal sets of functions which differ markedly from the already existing sets in the sense of continuity and valuedness over their domain of definition. Functions which are binary-valued, and discontinuous at zero-crossing points have been constructed. Included in this list are functions due to Haar, in 1910 [7]; Rademacher, 1922 [8]; and Walsh, 1923 [2]. Haar and Walsh functions are complete orthogonal sets, while the set of Rademacher functions is only orthogonal. It is easily shown that Rademacher functions form a subset of Walsh functions; yet, the two sets were constructed independently.

Of these binary functions, the set due to Walsh is more prominent in terms of recent studies and applications. Originally, Walsh defined his set on a half-open unit interval $[0,1)$. However, representations of these functions have been given on the interval $[-1/2,1/2)$. At a later stage, Paley [9] reintroduced Walsh functions to the scientific community by defining them as products of Rademacher functions. Walsh's definition seems more appealing to engineers because of the analogy with trigonometric functions in terms of ordering the functions according to the increasing average number of zero-crossings in a unit interval, called sequency. Paley's definition, on the other hand, seems attractive for analytical manipulations.

In 1949, Fine [10] published an excellent paper dealing with some mathematical properties of Walsh functions as defined earlier by Paley. In a following paper published in 1950 [11], Fine introduced the Walsh transform for representing nonperiodic functions. Further studies of generalized Walsh functions and transform were made later by Chrestenson [12], Selfridge [13] and others. Since then, many investigators have carried the work further, but mainly in the region of applications of Walsh theory.

3. Applications of Walsh Functions

Well-formulated applications of Walsh functions did not appear in the literature until the early 1960's with the appearance of the works by Corrington [14] and Weiser [15] who actually initiated the idea of employing Walsh functions in the studies of nonlinear problems.

Over the last few years, Walsh functions have been successfully applied to engineering, physics and other physical science problems. Their use in signal processing (audio and video) has been noticeably useful and decidedly advantageous in certain respects [5], [16]. Their use is by no means limited to these fields. Indeed, they have also been successfully applied in the medical area for purposes of ECG (electrocardiograms) and EEG (electroencephalograms) analysis, in the geological area for analysis of seismic data, and in the biological sciences for classification of biological forms.

With the realization of the equivalence of the discrete Walsh transform and the Hadamard transform, the use of Walsh functions has been even more favorable due to the ease of generating Hadamard matrices on digital computers. The discrete Walsh transform constitutes the basis for much of current investigations into the applications of Walsh functions. This is largely attributed to the ease and speed with which a fast Walsh transform (FWT) may be executed.

The most important factor in the increased use of Walsh functions is perhaps due to their digital nature. Another important factor is again due to the existence of simple software and hardware implementations of fast Walsh transform methods for the computations of digital or discrete time data.

As the Walsh transform matrix is purely real, with entries {+1, -1,}, fast operations require fewer operations than the comparable fast Fourier transform (FFT). This in turn implies savings in processing time and storage allocations when using a digital computer as the signal processor. This advantage is particularly important to researchers with limited facilities, a condition which is not uncommon to colleges and small-scale industries.

4. This Work

Admittedly, most applications of Walsh functions are in the area of digital signal processing; and thus the discrete aspect of Walsh theory has been dominant over the generalized case. However, the use of generalized Walsh functions and Walsh transform in signal analysis can also be utilized beneficially in a good number of applications.

This work is devoted primarily to studies in generalized Walsh theory and some of its significant applications in sequency spectral analysis of signals and systems. Our objective is the treatment of the Walsh theory in a unified presentation: concise and clear. The collection of research results in the covered areas should provide easy reference to concerned workers in the field. The applications, on the other hand, may motivate further investigations into uses of Walsh techniques, especially in the nonlinear area.

To these ends, it is hoped that this work will contribute to better understanding of Walsh theory and its applications, particularly in the analysis of stochastic problems [15], [17].

Selected References

1. Fourier, J.B., "Theorie Analytique de la Chaleur," 1828. English ed. "The Analytic Theory of Heat," 1878. Reprinted by Dover Pub. Co., New York, 1955.
2. Walsh, J.L., "A Closed Set of Orthogonal Functions," *Amer. J. Math.*, Vol. 44, pp. 5-24, 1923.
3. Anderson, T.W., *The Statistical Analysis of Time Series*, John Wiley & Sons, New York, N.Y., 1970.
4. Parzen, E., *Stochastic Processes*, Holden-Day, San Francisco, California, 1962.
5. Ahmed, N. and K.R. Rao, *Orthogonal Transforms for Digital Signal Processing*, Springer-Verlag, New York, N.Y., 1975.
6. Bell, W.W., *Special Functions for Scientists and Engineers*, Van Nostrand Co., London, 1968.
7. Haar, A., "Zur Theorie der Orthogonalen Funktionensysteme," *Math. Annalen*, Vol. 69, pp. 331-371, 1910.
8. Rademacher, H., "Einige Satze von allgemeinen Orthogonalfunktion," *Math. Annalen*, Vol. 87, pp. 122-138, 1922.
9. Paley, R.E.A.C., "A Remarkable Series of Orthogonal Functions," *Proc. London Math. Soc.*, Vol. 34, pp. 241-279, 1932.
10. Fine, N.J., "On the Walsh Functions," *Trans. Amer. Math.*, Vol. 65, pp. 373-414, 1949.
11. Fine, N.J., "The Generalized Walsh Functions," *Trans. Amer. Math. Soc.*, Vol. 69, pp. 66-77, 1950.
12. Chrestenson, H.E., "A Class of Generalized Walsh Functions," *Pacific J. Math.*, Vol. 5, pp. 17-31, 1955.
13. Selfridge, R.G., "A Generalized Walsh Transform," *Pacific J. Math.*, Vol. 5, pp. 451-480, 1955.
14. Corrington, M.S., "Advanced Analytical and Signal Processing Techniques," ASTIA Document No. AD-277942, April 1962.
15. Weiser, F.E., "Walsh Function Analysis of Instantaneous Nonlinear Stochastic Problems," Ph.D. Dissertation, Polytechnic Institute of New York, 1964.
16. Andrews, A.C., *Computer Techniques in Image Processing*, Academic Press, New York, N.Y., 1970.
17. Maqusi, M., "Walsh Analysis of Linear Dyadic-Invariant and Instantaneous Nonlinear Systems," Sc.D. Dissertation, New Mexico State University, Les Cruces, N.M., July 1973.

CHAPTER 2 · WALSH FUNCTIONS

1. Introduction

This chapter is primarily concerned with definitions of Walsh functions and presentation of their properties. Two standard methods of defining these functions are dealt with. The first method relies on a recursive-type relation and establishes Walsh functions in a form convenient for computational purposes. The second method defines Walsh functions as products of Rademacher functions. This latter approach generates a set which is useful for analytical considerations.

The concept of dyadic (or binary) representation of real numbers constitutes a significant aspect in the definition of Walsh functions and the derivation of their properties. Hence, a review of dyadic representation of real number appropriately precedes definitions of Walsh functions.

Upon defining Walsh functions as a complete orthogonal set, it becomes possible to employ these functions in Walsh series expansion representations for suitable functions. In this connection, we additionally discuss a useful method which sometimes facilitates the computations of Walsh expansion coefficients.

The development of a Walsh-Dirichlet kernel and associated properties forms another part of the chapter. In conclusion, the chapter presents a discussion on harmonic ordering of Walsh functions. This type of ordering divides the Walsh set into disjoint subsets. Members of these subsets are characterized as harmonically-related Walsh functions.

2. Dyadic Representation of Real Numbers

In this section we discuss dyadic representations of real numbers and the rules of dyadic addition [1].