U. Grenander

Applied Mathematical Sciences 33

Regular Structures

Lectures in Pattern Theory Volume III

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PREFACE

Most of the material in this book has been presented in lectures at Brown University, either in courses taught in the Division of Applied Mathematics or in the author's Research Seminar in Pattern Theory. I would like to thank the several members of the Division of Applied Mathematics that have participated in the discussions and in particular W. Freiberger, S. Geman, C.-R. Hwang, D. McClure and P. Thrift.

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Providence, Rhode Island
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INTRODUCTION

This is the third and final volume of the Lectures in Pattern Theory. Its two first chapters describe the science-theoretic principles on which pattern theory rests. Chapter 3 is devoted to the algebraic study of regularity while Chapter 5 contains new results in metric pattern theory. Some brief remarks on topological image algebras can be found in Chapter 4.

Two chapters deal with pattern synthesis: Chapter 6 on scientific hypothesis formation and Chapter 7 on social domination structures. In Chapter 8 we study taxonomic patterns, both their synthesis and analysis, while in the last chapter we investigate a pattern processor for doing semantic abduction.

The material contained in the three volumes has been presented in historical rather than logical order. A reader approaching pattern theory for the first time is advised to do it in the following order,

Introduction to regular structures	{ Chapters 1 and 2 of Volume III.

Pattern synthesis	Chapters 1,2,3 of Volume I Chapters 3,4,5 of Volume III Chapter 4 of Volume I Chapters 6,7 of Volume III
Pattern analysis	Chapters 1,2,3,4,5 of Volume II Chapter 8 of Volume III
Pattern processors	Chapters 6,7 of Volume II Chapter 9 of Volume III

Most of the content is due to the author and the members of the Research Seminar in Pattern Theory at Brown University. With a few exceptions it has not appeared in print before.

Space does not permit the inclusion of all the new results. So for example have we not included the analysis of star-shaped patterns and of spectroscopic patterns, nor the study of growth patterns based on contact transformations. The method of sieves, developed for pattern inference, will be presented in the author's forthcoming book, "Abstract Inference". A separate publication will also appear containing mathematical software that we have written for the computational experiments that have played an important role during the growth of pattern theory.

Lord Kenneth Clark once described the publication of lectures as "a well-known form of literary suicide". One can certainly argue against publishing lecture notes since they are likely to contain obscurities and mistakes and be too fragmented to offer a complete view of the subject.

In spite of this we decided to publish these Notes rather than to wait for a polished and complete presentation. As mentioned in the Introduction to Volume I a more definitive version will appear eventually. In the meantime these three volumes with all their imperfections will have to suffice.

CHAPTER 1
PATTERNS: FROM CHAOS TO ORDER

1.1. The search for regularity

The search for regularity is a dominant theme in man's attempt to understand the world around him. Any such attempt is based on an assumption, tacitly made or explicit, that phenomena in nature and in the man-made world are governed by laws that result in order and structure.

Or to quote Hume in his Treatise of Human Understanding, Book I, Sect. VI: "If reason determined us, it would proceed upon that principle, *that instances, of which we have had no experience, must resemble those, of which we have had experience, and that the course of nature continues always uniformly the same.*" This principle underlies the incomplete inductive reasoning used in science as well as in everyday life.

Indeed, it is hard to see how anything could be really understood in a completely chaotic world, where events followed each other in an arbitrary fashion, where chaos reigned and no rules restricted what could occur. It would be impossible to plan for the future, even to take action to make the individual or the species survive in such a frightening and mysterious environment.

Already in pre-scientific times man must have tried to find regularities that he could rely on in his everyday life or that would give him a feeling of security in a hostile world. Or, quoting from Frazer's Chapter LXIX of "The Golden Bough",

> "In magic, man depends on his own strength to meet the difficulties and dangers that beset him from every side. He believes in a certain established order of nature on which he can surely count, and which he can manipulate for his own ends. When he discovers his mistake, when he recognizes sadly that both the order of nature which he had assumed and the control which he had believed himself to exercise over it were purely imaginary, he ceases to rely on his own intelligence and his own unaided efforts, and throws himself humbly on the mercy of certain great invisible beings behind the veil of nature, to whom he now ascribes all those far-reaching powers which he once arrogated to himself" ...

Magic is superseded by a religious belief in gods -

> "But as time goes on this explanation in its turn proves to be unsatisfactory. For it assumes that the succession of natural events is not determined by immutable laws, but is to some extent variable and irregular, and this assumption is not borne out by closer observation. On the contrary, the more we scrutinize that succession the more we are struck by the rigid uniformity, the punctual precision with which, wherever we can follow them, the operators of nature are carried on."

Most sciences pass through an early stage of collecting isolated data, assembling curious objects or facts. Already taxonomic attempts to classify objects or facts represent a tendency towards generality and "immutable laws" and "rigid uniformity". At a somewhat later stage, usually overlapping with the earlier one, one strives for the explicit formulation of general principles. The scientist's rule is not just to discover or invent such principles, but it is at least as much concerned with the logical analysis of them and to deduce consequences. It depends upon the consequences and their relation to the observed world how successful the scientist has been in describing the regularities.

Viewed from our own time and in a more abstract setting such attempts could be formalized as *formal systems:* certain basic statements or procedures and rules how to apply them in order to explain certain phenomena. For example, statement A implies B, another statement C implies A or, formally

$$\begin{cases} A \rightarrow B \\ C \rightarrow A \end{cases} \tag{1.1}$$

In pre-Galilean mechanics A could be "object 1 is heavier than object 2", B="object 1 falls faster than object 2", and C="objects 1 and 2 have the same volume, the first is made of lead and the other of iron".

For a given set of basic statements (1.1) the richness of the results of applying rules will depend upon how sophisticated are the syllogisms to be used. If the usual rules of logic are applied one gets as consequences of the statements in (1.1) if B does not occur A cannot hold, if C is true then B must hold, etc:

$$\begin{cases} \sim B \rightarrow \sim A \\ C \rightarrow B \\ \cdots \end{cases} \tag{1.2}$$

In order that a system describing regularity deserve its name it must have some permanence in time and space. If it only applies to a particular time and a particular place it is a *datum*, an isolated observation, but not a law of nature. Therefore one must insist that the statements should be true in some generality.

When we speak of laws, order, patterns, we are concerned with more than isolated facts. Laws deal with several alternatives, interesting laws with a great number of alternatives. We therefore have to adopt an *ensemble* attitude: the pattern should refer to an ensemble of possible cases. In such an ensemble order is viewed as the uniform validity of certain properties. This is still rather vague but will become more precise when we examine a number of regular structures in Section 1.2.

The symbols used (A,B,...) are irrelevant, we could equally well have employed other abbreviations for the statements. We could express this by saying that we are thinking of a particular interpretation of the formal statement (1.1) and the interpretation is fixed while the formalization of it remains arbitrary to some extent. One and the same regular structure could be expressed through many formal systems, mutually equivalent. As long as the formulas mean the same we have no reason to prefer one before the other unless we bring in other criteria based on notions such as simplicity and convenience.

From a formal point of view we need not distinguish between statements like $A \to B$ and syllogisms like $(x \to y) \to (\sim y \to \sim x)$. In the interpretation used above the first one was based on *empirical knowledge* while the second one was an *analytical truth*. Formally they can both be viewed as laws or axioms that we can combine together to arrive at other, derived statements. The number of derived statements can be large, even infinite.

To bring out more clearly the conceptual structure of this kind of regularity, let us consider another case, a fragment of Newtonian mechanics for point masses. We would then have statements like

$$\begin{cases} m_1 & = \ldots; m_2 = \ldots; \ \ldots \ x_1 = \ldots, y_1 = \ldots; \ \ldots \\ F_{gr} & = k\,\dfrac{m_1 m_2}{r^2}; \\ k & = \ldots; \\ r & = \sqrt{(x_1-x_2)^2+(y_1-y_2)^2+(z_1-z_2)^2} \\ m\ddot{x} & = F^x; m\ddot{y} = F^y, \ldots; \\ F_{12} & = -F_{21}; \\ F^x_{fr} & = -f\dot{x}; \\ f & = \ldots \end{cases} \tag{1.3}$$

together with the other statements representing calculus and syllogisms. Combining statements together in a "meaningful" manner we can derive other statements and describe, analyze, and predict the behavior of mechanical phenomena. In other words, we can express the regularities of such phenomena.

In (1.3) the natural invariances are the invariances with respect to Galilean transformations.

$$\begin{cases} t = t' \\ x = x' + at \\ y = y' + bt \\ z = z' + ct \end{cases} \tag{1.4}$$