INTRODUCTORY COLLEGE MATHEMATICS

HACKWORTH
and
HOWLAND

SAUNDERS SERIES IN MODULAR MATHEMATICS

Algebra 1

INTRODUCTORY COLLEGE MATHEMATICS

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PREFACE

Algebra 1

This book is one of the sixteen content modules in the Saunders Series in Modular Mathematics. The modules can be divided into three levels, the first of which requires only a working knowledge of arithmetic. The second level needs some elementary skills of algebra and the third level, knowledge comparable to the first two levels. *Algebra 1* is in level 1. The groupings according to difficulty are shown below.

Level 1	Level 2	Level 3
Tables and Graphs	*Numeration*	*Real Number System*
Consumer Mathematics	*Metric Measure*	*History of Real Numbers*
Algebra 1	*Probability*	*Indirect Measurements*
Sets and Logic	*Statistics*	*Algebra 2*
Geometry	*Geometric Measures*	*Computers*
		Linear Programming

The modules have been class tested in a variety of situations: large and small discussion groups, lecture classes, and in individualized study programs. The emphasis of all modules is upon ideas and concepts.

Algebra 1 is a necessity for all non math-science students who need a basic understanding of the rational numbers and a skill in solving linear equations. *Algebra 1* is not a complete first course in algebra, but math-science majors may find it useful for review purposes.

Algebra 1 begins by developing skills in the fundamental operations with signed numbers and rational numbers. Methods of evaluating and simplifying expressions precede the development of equation solving techniques. The solving of word problems is also included.

In preparing each module we have been greatly aided by the valuable suggestions of the following excellent reviewers: William Andrews, Triton College, Ken Goldstein, Miami-Dade Community College, Don Hostetler, Mesa Community College, Karl Klee, Queensboro Community College, Pamela Matthews, Chabot College, Robert Nowlan, Southern Connecticut State College, Ken Seydel, Skyline College, Ara Sullenberger, Tarrant County Junior College, and Ruth Wing, Palm Beach Junior College. We thank them, and the staff at W. B. Saunders Company for their support.

Robert D. Hackworth
Joseph W. Howland

NOTE TO THE STUDENT

OBJECTIVES

Upon completion of this module, the reader is expected to be able to demonstrate the following skills:

1. Add, subtract, multiply, and divide signed numbers.

2. Add, subtract, multiply, and divide rational numbers.

3. Find reciprocals and opposites for rational numbers.

4. Evaluate signed number expressions.

5. Simplify algebraic expressions.

6. Solve and check equations of the form $ax = b$, $ax + b = c$ and equations with the variable in both parts of the equation.

7. Solve literal equations and formulae.

8. Solve simple word problems.

Three types of problem sets with answers are used in this module. Progress Tests are always short. Questions asked on each Progress Test come from the section immediately preceding it.

Exercise Sets involve more problems and cover a greater number of concepts than the Progress Tests. Section I problems in the Exercise Sets are keyed directly to the module objectives. Section II problems of the Exercise Sets are Challenge Problems.

A Self-Test is found at the end of the module. The Self-Test problems are representative of the entire module.

In learning the material, the student is encouraged to try each problem set as it is encountered, check answers and restudy those sections where difficulties are discovered. This procedure is guaranteed to be both efficient and effective.

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ALGEBRA 1

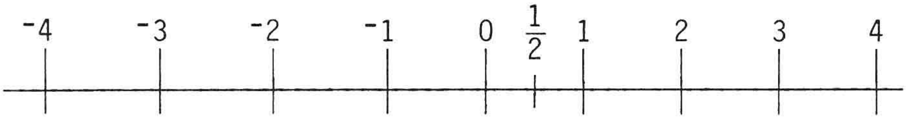
INTRODUCTION

Algebra is the language of mathematics. College mathematics, science, business, economics, psychology, etc. are often taught and explained in the language of mathematics -- algebra. The reader of this material should find the module of immediate benefit in every other course that requires or assumes some background in mathematics.

The emphasis of this module is on the skills of working with signed numbers, fractions, algebraic expressions, and equations.

THE SIGNED NUMBERS

On the number line below, zero and the numbers to the right of zero, including the fractions and whole numbers, are sometimes called the numbers of arithmetic. The numbers of arithmetic are commonly associated with ideas of quantity or distance. For example, $\frac{1}{2}$ and 5 are numbers of arithmetic that might be naturally used to describe the quantity $\frac{1}{2}$ pound of sugar or the distance 5 miles.



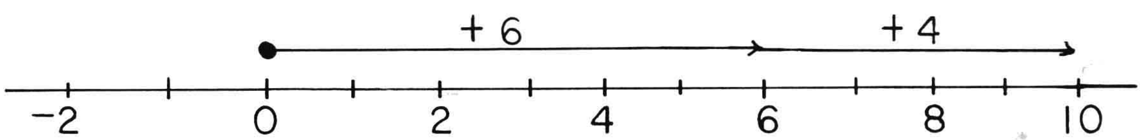
The numbers to the left of 0 on the number line are called the negative numbers. Unlike the numbers of arithmetic, the negative numbers are not commonly associated with ideas of quantity.

It doesn't sound natural to refer to a negative $\frac{1}{2}$ pound of sugar or a negative 5 miles. Nevertheless, the negative numbers are most useful in algebra and a skill at adding, subtracting, multiplying, and dividing the negative numbers and the arithmetic numbers is necessary for many algebra problems encountered in business, industry, and science.

Sometimes the numbers on the number line are called signed numbers. The numbers to the right of 0 may be indicated by a plus sign and are called positive numbers. The numbers to the left of 0 may be indicated by a raised dash or a minus sign and are called negative numbers. The number 0 is neither positive nor negative.

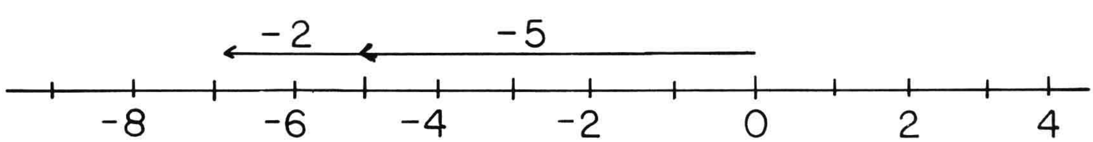
Another name for the signed numbers is directed numbers. Numbers on the number line are called directed numbers because each number represents both a distance from zero and a direction from zero. For example, the directed number $^+7$ represents the distance 7 from 0 and its direction is to the right of 0. The directed number $^-3$ represents a distance of 3 from 0 and its direction is to the left of 0. If a number does not have a sign, it is considered positive. $3 = {}^+3$.

The addition of signed numbers can be understood in terms of the distances and directions of the numbers. For example, the sum of $^+6$ and $^+4$ may be illustrated by the number line figure below.



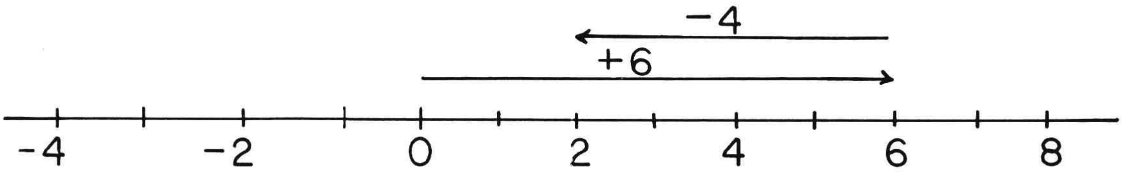
The sum of $^+6$ and $^+4$ is $^+10$ because the distances are 6 and 4 and both directions are to the right.

To add $^-5$ and $^-2$ the sum may be illustrated by the number line figure below.



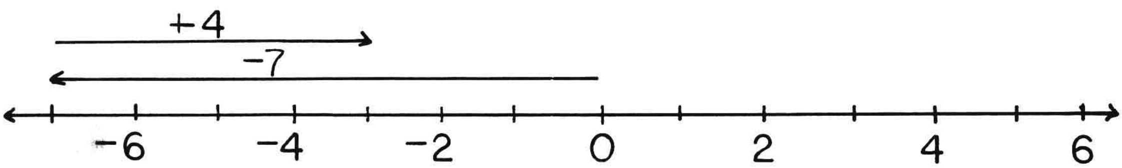
The sum of $^-5$ and $^-2$ is $^-7$ because the distances are 5 and 2 and both directions are to the left.

To add $^{+}6$ and $^{-}4$ the sum may be illustrated by the number line figure shown below.



The sum of $^{+}6$ and $^{-}4$ is $^{+}2$ because the distance 6 is to the right, but the distance 4 is to the left. The combination of these two directed distances is $^{+}2$.

To add $^{-}7$ and $^{+}4$ the sum may be illustrated by the figure shown below.



The sum of $^{-}7$ and $^{+}4$ is $^{-}3$ because the distance 7 is to the left and the distance 4 is to the right. The result is a combination of the two directed distances that ends at $^{-}3$.

The rules for adding any two signed numbers are:

1. If the numbers have the same sign, add the distances and retain the sign.

2. If the numbers have opposite signs, subtract the smaller distance from the larger and use the sign of the larger distance.

3. If one of the numbers is zero then the answer is the other number.

Some examples of correctly added signed numbers are shown below. Notice that a number is considered to be positive if the plus sign is omitted.

$(^{+}8) + (^{-}7) = 1$ $\qquad$ $(^{-}16) + (^{-}2) = {^{-}18}$

$(^{-}9) + (4) = {^{-}5}$ $\qquad$ $(11) + (5) = 16$

$(^{-}7) + 7 = 0$ $\qquad$ $(8) + (^{-}8) = 0$

Subtraction of signed numbers involves a change of direction for the number being subtracted. In the problem 5 - 3 the 5 is called the minuend and the 3 is called the subtrahend. When a subtraction problem is written in vertical columns, the top number is the minuend and the bottom number is the subtrahend.

943	Minuend
-658	Subtrahend

The subtraction of signed numbers has an interesting relationship to the addition of signed numbers. This relationship may be seen in the columns of examples below. The left column below shows subtraction. The right column shows addition. What is the relation between the two sets of problems?

$4 - (1)$	$4 + (^{-}1)$
$5 - (7)$	$5 + (^{-}7)$
$^{-}4 - (3)$	$^{-}4 + (^{-}3)$
$^{-}5 - (^{-}2)$	$^{-}5 + (^{+}2)$
$6 - (^{-}3)$	$6 + (^{+}3)$

Each pair above is equivalent because they have the same evaluation. For example, $4 - 1 = 3$ and $4 + (^{-}1) = 3$. $6 - (^{-}3) = 9$ and $6 + (^{+}3) = 9$. Because of the relationship shown above, a rule for subtraction can be made.

The rule for signed number subtraction is: Change the sign of the subtrahend and then proceed with the problem as signed number addition. Four more examples of signed number subtraction are shown below.

$$8 - (5) = 8 + (^{-}5) = 3$$

$$^{-}10 - (6) = {}^{-}10 + (^{-}6) = {}^{-}16$$

$$12 - (19) = 12 + (^{-}19) = {}^{-}7$$

$$^{-}5 - (^{-}11) = {}^{-}5 + (11) = 6$$

The fact that some subtraction problems such as $6 - (^{-}5) = 11$ look like additions and some additions such as $^{-}6 + 13 = 7$ look like subtractions creates problems for students who ask, "How do you know whether to add or subtract?" The best advice for such students is to simply follow the directions of the signs without undue concern about which operation is being performed.

For example, consider the following problem involving seven numbers.

$$8 - 9 + 7 + 3 - 6 - 4 + 10$$

To do the problem, think of each number as including the sign that appears directly to its left. Circles are drawn around the numbers and their signs in the line below.

(8) (- 9) (+ 7) (+ 3) (- 6) (- 4) (+ 10)

The problem can now be correctly done by treating the numbers as $^{+}8$, $^{-}9$, $^{+}7$, $^{+}3$, $^{-}6$, $^{-}4$, and $^{+}10$. Without thinking add or subtract, go right 8, left 9, right 7, right 3, left 6, left 4, and right 10. The result is $^{+}9$. $^{+}9$ is called the evaluation of the problem above.

Progress Test 1

1. Add: $5 + {}^{-}14 =$

   ${}^{-}7 + {}^{-}6 =$

2. Subtract: $4 - ({}^{-}11) =$

   ${}^{-}15 - (9) =$

3. Evaluate: $5 - 4 - 8 + 2 =$

4. Evaluate: ${}^{-}5 - 6 + 9 + 13 - 20 + 4 =$

5. Evaluate: $6 - 7 + 4 + 5 + 7 - 3$

6. Evaluate: ${}^{-}3 - 1 + 3 + 5 + {}^{-}3 - 8$

MULTIPLYING AND DIVIDING SIGNED NUMBERS

Multiplication is frequently explained in terms of repeated addition. For example, the problem 8 • 3 can be interpreted as the addition of eight 3's. Since the addition of eight 3's is

24, the multiplication of the signed numbers $^{+}8$ and $^{+}3$ can be reasonably expected to be $^{+}24$.

Less obvious, perhaps, is the correct answer for $^{+}5$ times $^{-}4$. Interpreting the multiplication of $5 \cdot {}^{-}4$ as an addition problem results in the addition of five $^{-}4$'s.

$$^{-}4 + {}^{-}4 + {}^{-}4 + {}^{-}4 + {}^{-}4 = {}^{-}20$$

Since the addition of five $^{-}4$'s is $^{-}20$, the multiplication of $5 \cdot {}^{-}4$ must also be $^{-}20$.

An addition interpretation for the multiplication of $^{-}6$ and 3 poses a problem because it suggests that 3 should be used as an addend $^{-}6$ times. How is something added a negative number of times? The difficulty is overcome with the following considerations:

1. In arithmetic the product of two numbers is the same regardless of which number is first and which is second. For example, $2 \cdot 5$ has the same answer as $5 \cdot 2$. This property of numbers is called the Commutative Property of Multiplication.

2. The flexibility of interchanging two numbers in a multiplication problem should be extended to the multiplication of signed numbers.

3. Consequently, ${}^{-}6 \cdot 3$ should have the same answer as $3 \cdot {}^{-}6$. The answer to $3 \cdot {}^{-}6$ is $^{-}18$ because it can be interpreted as ${}^{-}6 + {}^{-}6 + {}^{-}6$. Since $3 \cdot {}^{-}6 = {}^{-}18$ then ${}^{-}6 \cdot 3$ must also equal $^{-}18$.

By interpreting multiplication of signed numbers as addition, examples have been shown to support the following rules:

1. The product of two positive numbers is a positive number.

2. The product of a positive and a negative, regardless of which is first or second, is negative.

A third possibility cannot be explained by signed number addition. That possibility is the product of two negative numbers such as $^{-}5$ times $^{-}2$. Before explaining why ${}^{-}5 \cdot {}^{-}2$ is $^{+}10$, two prior concepts must be established.

1. Any number times 0 is 0. This fact is not new or startling in itself, but it is interesting that it is necessary to the justification that the product of two negatives is a positive.

2. Two expressions like $6 \cdot (3+2)$ and $(6 \cdot 3) + (6 \cdot 2)$ are related because they involve the same numbers, but the parentheses indicate that addition is performed first in $6 \cdot (3+2)$ but multiplication is performed first in $(6 \cdot 3) + (6 \cdot 2)$.

$6 \cdot (3+2)$	$(6 \cdot 3) + (6 \cdot 2)$
$6 \cdot 5$	$18 + 12$
30	30

It is no coincidence that the answers are the same. Whenever two expressions are related like $6 \cdot (3+2)$ and $(6 \cdot 3) + (6 \cdot 2)$ the answers will be the same. This property of numbers is called the Distributive Property of Multiplication over Addition.

Using the ideas that $^{-}5 \cdot 0$ should be 0 and $^{-}5\ (^{-}2 + 2)$ should have the same answer as $(^{-}5 \cdot {}^{-}2) + (^{-}5 \cdot 2)$ it is now possible to show that $^{-}5 \cdot {}^{-}2$ must be equal to $^{+}10$

$$
\begin{aligned}
^{-}5 \cdot {}^{-}2 &= (^{-}5 \cdot {}^{-}2) + 0 \\
&= (^{-}5 \cdot {}^{-}2) + (^{-}10 + 10) \\
&= (^{-}5 \cdot {}^{-}2) + (^{-}5 \cdot 2) + 10 \\
&= {}^{-}5 \cdot (^{-}2 + 2) + 10 \\
&= {}^{-}5 \cdot 0 + 10 \\
&= 0 + 10 \\
&= 10
\end{aligned}
$$

The argument above shows why $^{-}5 \cdot {}^{-}2$ equals $^{+}10$. It is based on the desirability of retaining other number properties enjoyed by the arithmetic numbers. Similar arguments can be made for $^{-}6 \cdot {}^{-}3 = {}^{+}18$ and $^{-}7 \cdot {}^{-}9 = {}^{+}63$.

By example, the following rules for multiplying signed numbers have been shown:

1. When multiplying signed numbers with the same sign, the answer is positive.

2. When multiplying signed numbers with different signs, the answer is negative.

3. When multiplying any signed number by 0, the answer is 0.

A chart showing the rules for multiplying signed numbers is shown below.

First Number	Second Number	Product
+	+	+
+	-	-
-	+	-
-	-	+

Three numbers are involved in every multiplication; the two numbers being multiplied and the answer. Looking closely at the table, it can be seen that there are always an even number of negative numbers in a multiplication. The first row shows no negatives and the last three rows show two negatives. This idea may be expanded to a multiplication expression involving three or more signed numbers. It provides a quick, easy check for every signed number multiplication. Count all the negative signs in the problem and the answer; the result must be an even number.

The rules for division of positive and negative numbers are similar to those for multiplication. This is most easily seen by considering the check of a division problem.

$$12 \overline{)\,84}^{\;7}$$

To check this division, 12 is multiplied by 7. Since 12 times 7 equals 84, the division is correct.

The check for any division problem is a multiplication. Since the multiplication of signed numbers always must result in using an even number of negative signs, the same restriction is true for division. Consequently, the following rules apply for signed number division.

1. In dividing, if the numbers have the same sign the answer is positive.

2. In division, if the numbers have different signs, the answer is negative.

3. If 0 is divided by any number except 0 the answer is 0.

4. Zero may never be divided into any number, including itself.

Below are shown some correctly worked examples of signed number multiplication and division.

$^-8 \cdot 7 = {}^-56$ $\qquad$ $^-16 \div {}^-2 = 8$

$^-4 \cdot {}^-9 = 36$ $\qquad$ $28 \div {}^-7 = {}^-4$

$^-2 \cdot 11 = {}^-22$ $\qquad$ $\frac{^-14}{2} = {}^-7$

$5 \cdot {}^-2 \cdot {}^-3 \cdot {}^-1 \cdot 2 = {}^-60$ $\qquad$ $\frac{^-63}{^-7} = 9$

Progress Test 2

1. $17 \cdot {}^-4 = ?$

2. $^-51 \cdot {}^-2 = ?$

3. $^-48 \div 4 = ?$

4. $\frac{^-24}{^-6} = ?$

5. $^-4 \cdot {}^-2 \cdot {}^-1 \cdot 2 \cdot {}^-3 = ?$

6. $\frac{^-36}{0} = ?$

Exercise Set 1

I. Perform the indicated operations.

1. ${}^{-}5 + {}^{-}13 =$
2. ${}^{-}8 + {}^{-}3 =$
3. ${}^{-}8 + 2 =$
4. $3 + {}^{-}7 =$
5. $7 + {}^{-}3 =$
6. ${}^{-}7 + {}^{-}3 =$
7. $5 + {}^{-}5 =$
8. ${}^{-}6 + 6 =$
9. $21 + {}^{-}19 =$
10. ${}^{-}37 + {}^{-}38 =$
11. ${}^{-}9 + 2 =$
12. $37 + {}^{-}6 =$
13. ${}^{-}17 + 17 =$
14. $5 - 2 =$
15. $6 - 7 =$
16. $8 - {}^{-}8 =$
17. ${}^{-}3 - 1 =$
18. $7 - 31 =$
19. $26 - 2 =$
20. $5 - {}^{-}9 =$
21. ${}^{-}3 - 2 =$
22. ${}^{-}5 - {}^{-}7 =$
23. ${}^{-}15 - {}^{-}1 =$
24. ${}^{-}8 - 3 =$
25. $6 - 1 =$
26. $39 - 58 =$
27. $5 - 2 - 3 + 7 =$
28. ${}^{-}7 + 6 + 7 - 3 =$
29. ${}^{-}6 - 4 - 2 - 13 - 2 =$
30. $10 + 3 + 2 - 3 - 2 - 10 =$
31. ${}^{-}41 - 2 - 3 + 7 + 41 =$
32. $19 + 2 - 21 - 3 + 7 + 10 =$
33. ${}^{-}2 + 2 + 17 - 26 + 19 =$
34. $6 - 13 + 5 - 7 - 9 - 10 =$

35. $2 \cdot {}^{-}3 =$

36. ${}^{-}5 \cdot 2 =$

37. ${}^{-}3 \cdot 2 =$

38. $2 \cdot {}^{-}5 =$

39. ${}^{-}6 \cdot {}^{-}7 =$

40. $4 \cdot {}^{-}19 =$

41. ${}^{-}6 \cdot {}^{-}13 =$

42. ${}^{-}5 \cdot {}^{-}2 =$

43. ${}^{-}5 \cdot 3 =$

44. $4 \cdot {}^{-}6 =$

45. ${}^{-}43 \cdot {}^{-}1 =$

46. $0 \cdot {}^{-}8 =$

47. ${}^{-}1 \cdot 6 =$

48. ${}^{-}13 \cdot 0 =$

49. $24 \div {}^{-}2 =$

50. ${}^{-}24 \div 2 =$

51. ${}^{-}24 \div {}^{-}2 =$

52. $24 \div 2 =$

53. ${}^{-}36 \div {}^{-}36 =$

54. $13 \div {}^{-}1 =$

55. $\frac{{}^{-}1}{{}^{-}1} =$

56. $\frac{{}^{-}8}{{}^{-}2} =$

57. $\frac{10}{{}^{-}5} =$

58. $\frac{{}^{-}14}{7} =$

59. $\frac{{}^{-}8}{{}^{-}1} =$

60. $\frac{91}{1} =$

II. Challenge Problems

1. ${}^{-}5 - {}^{-}2 - {}^{-}3 - 49 = ?$

2. $360 \div {}^{-}2 \div {}^{-}1 \div {}^{-}18 \div 2 = ?$

3. ${}^{-}1200 \div 3 \div {}^{-}8 \div 25 \div {}^{-}2 \div {}^{-}1 = ?$

4. $\dfrac{\dfrac{{}^{-}35}{{}^{-}7}}{5} \div \dfrac{\dfrac{{}^{-}6}{2}}{\dfrac{{}^{-}1}{3}} =$