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Spectral Theory and Geometric Analysis

International Conference in
Honor of Mikhail Shubin's 65th Birthday
Spectral Theory and Geometric Analysis
July 29–August 2, 2009
Northeastern University, Boston, MA

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Spectral Theory and Geometric Analysis

Preface

Misha Shubin made many seminal contributions to Spectral Theory and Geometric Analysis. He is also an outstanding teacher: he directed nearly twenty Ph.D. dissertations, and influenced many young mathematicians who were not his students. His book Pseudodifferential Operators and Spectral Theory, written more than 30 years ago, is still a standard textbook.

Mikhail Shubins 65th Birthday was celebrated at a conference titled *Spectral Theory and Geometric Analysis* held at Northeastern University in Boston in the summer of 2009. The speakers at this conference were leading mathematicians working in Global Analysis. The call for papers for this volume went to all participants of the conference.

We would like to thank the authors who contributed to this volume as well as those who served as referees.

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Leonid Friedlander
Thomas Kappeler
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Resolution of smooth group actions

Pierre Albin and Richard Melrose

ABSTRACT. A refined form of the 'Folk Theorem' that a smooth action by a compact Lie group can be (canonically) resolved, by iterated blow up, to have unique isotropy type is proved in the context of manifolds with corners. This procedure is shown to capture the simultaneous resolution of all isotropy types in a 'resolution structure' consisting of equivariant iterated fibrations of the boundary faces. This structure projects to give a similar resolution structure for the quotient. In particular these results apply to give a canonical resolution of the radial compactification, to a ball, of any finite dimensional representation of a compact Lie group; such resolutions of the normal action of the isotropy groups appear in the boundary fibers in the general case.

Introduction

Borel showed that if the isotropy groups of a smooth action by a compact Lie group, G, on a compact manifold, M, are all conjugate then the orbit space, $G\backslash M$, is smooth. Equivariant objects on M, for such an action, can then be understood directly as objects on the quotient. In the case of a free action, which is to say a principal G-bundle, Borel showed that the equivariant cohomology of M is then naturally isomorphic to the cohomology of $G\backslash M$. In a companion paper, [**1**], this is extended to the unique isotropy case to show that the equivariant cohomology of M reduces to the cohomology of $G\backslash M$ with coefficients in a flat bundle (the Borel bundle). In this paper we show how, by resolution, a general smooth compact group action on a compact manifold is related to an action with unique isotropy type on a resolution, canonically associated to the given action, of the manifold to a compact manifold with corners.

The resolution of a smooth Lie group action is discussed by Duistermaat and Kolk [**7**] (which we follow quite closely), by Kawakubo [**11**] and by Wasserman [**13**] but goes back at least as far as Jänich [**10**], Hsiang [**9**], and Davis [**6**]. See also the discussion by Brüning, Kamber and Richardson [**5**] which appeared after the present work was complete. In these approaches there are either residual finite group actions, particularly reflections, as a consequence of the use of real projective blow up or else the manifold is repeatedly doubled. Using radial blow up, and hence working in the category of manifolds with corners, such problems do not arise.

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For a general group action, M splits into various isotropy types

$$M^{[K]} = \{\zeta \in M : G_\zeta \text{ is conjugate to } K\},\ G_\zeta = \{g \in G : g\zeta = \zeta\},\ \zeta \in M.$$

These are smooth manifolds but not necessarily closed and the orbit space is then in general singular. We show below that each $M^{[K]}$ has a natural compactification to a manifold with corners, $Y_{[K]}$, the boundary hypersurfaces of which carry equivariant fibrations with bases the compactifications of the isotropy types contained in the closure of $M^{[K]}$ and so corresponding to larger isotropy groups. Each fiber of these fibrations is the canonical resolution of the normal action of the larger isotropy group. These fibrations collectively give what we term a *resolution structure*, $\{(Y_I, \phi_I); I \in \mathcal{I}\}$, the index set being the collection of conjugacy classes of isotropy groups, i.e. of isotropy types, of the action. If M is connected there is always a minimal 'open' isotropy type $\mu \in \mathcal{I}$, for which the corresponding manifold, $Y_\mu = Y(M)$, (possibly not connected) gives a resolution of the action on M. That is, there is a smooth G-action on $Y(M)$ with unique isotropy type and a smooth G-equivariant map

$$\beta : Y(M) \longrightarrow M \tag{1}$$

which is a diffeomorphism of the interior of $Y(M)$ to the minimal isotropy type. Here, β is the iterated blow-down map for the resolution. There is a G-invariant partition of the boundary hypersurfaces of $Y(M)$ into non-self-intersecting collections H_I, labelled by the non-minimal isotropy types $I \in \mathcal{I} \setminus \{\mu\}$, and carrying G-equivariant fibrations

$$\phi_I : H_I \longrightarrow Y_I. \tag{2}$$

Here Y_I resolves the space M_I, the closure of the corresponding isotropy type M^I,

$$\beta_I : Y_I \longrightarrow M_I,\ \beta\big|_{H_I} = \beta_I \circ \phi_I. \tag{3}$$

Thus the inclusion relation between the M_I corresponding to the stratification of M by isotropy types, is 'resolved' into the intersection relation between the H_I. The resolution structure for M, thought of as the partition of the boundary hypersurfaces with each collection carrying a fibration, naturally induces a resolution structure for each Y_I. Since the fibrations are equivariant the quotients Z_I of the Y_I by the group action induce a similar resolution structure on the quotient $Z(M)$ of $Y(M)$ which resolves the quotient, the orbit space, $G\backslash M$.

As noted above, in a companion paper [**1**], various cohomological consequences of this construction are derived. The 'lifts' of both the equivariant cohomology and equivariant K-theory of a manifold with a group action to its resolution structure are described. These lifted descriptions then project to corresponding realizations of these theories on the resolution structure for the quotient. As a consequence of the forms of these resolved and projected theories a 'delocalized' equivariant cohomology is defined, and shown to reduce to the cohomology of Baum, Brylinski and MacPherson in the Abelian case in [**3**]. The equivariant Chern character is then obtained from the usual Chern character by twisting with flat coefficients and establishes an isomorphism between equivariant K-theory with complex coefficients and delocalized equivariant cohomology. Applications to equivariant index theory will be described in [**2**].

For the convenience of the reader a limited amount of background information on manifolds with corners and blow up is included in the first two sections. The abstract notion of a resolution structure on a manifold with corners is discussed in §3 and the basic properties of G-actions on manifolds with corners are described in §4. The standard results on tubes and collars are extended to this case in §5. In §6 it is shown that for a general action the induced action on the set of boundary hypersurfaces can be appropriately resolved. The canonical resolution itself is then presented in §7, including some simple examples, and the induced resolution of the orbit space is considered in §8. Finally §9 describes the resolution of an equivariant embedding and the 'relative' resolution of the total space of an equivariant fibration.

The authors are grateful to Eckhard Meinrenken for very helpful comments on the structure of group actions, and to an anonymous referee for remarks improving the exposition.

1. Manifolds with corners

By a *manifold with corners,* M, we shall mean a topological manifold with boundary with a covering by coordinate charts

$$M = \bigcup_j U_j,\ F_j : U_j \longrightarrow U_j' \subset \mathbb{R}^{m,\ell} = [0,\infty)^\ell \times \mathbb{R}^{m-\ell}, \tag{1.1}$$

where the U_j and U_j' are (relatively) open, the F_j are homeomorphisms and the transition maps

$$F_{ij} : F_i(U_i \cap U_j) \longrightarrow F_j(U_i \cap U_j),\ U_i \cap U_j \neq \emptyset \tag{1.2}$$

are required to be smooth in the sense that all derivatives are bounded on compact subsets; an additional condition is imposed below. The ring of smooth functions $\mathcal{C}^\infty(M) \subset \mathcal{C}^0(M)$ is fixed by requiring $(F_j^{-1})^*(u\big|_{U_j})$ to be smooth on U_j', in the sense that it is the restriction to U_j' of a smooth function on an open subset of $\mathbb{R}^m$.

The part of the boundary of smooth codimension one, which is the union of the inverse images under the F_i of the corresponding parts of the boundary of the $\mathbb{R}^{m,\ell}$, is dense in the boundary and the closure of each of its components is a *boundary hypersurface* of M. More generally we shall call a finite union of non-intersecting boundary hypersurfaces a *collective boundary hypersurface.* We shall insist, as part of the definition of a manifold with corners, that these boundary hypersurfaces each be *embedded,* meaning near each point of each of these closed sets, the set itself is given by the vanishing of a local smooth defining function x which is otherwise positive and has non-vanishing differential at the point. In the absence of this condition M is a *tied manifold.* It follows that each collective boundary hypersurface, H, of a manifold with corners is globally the zero set of a smooth, otherwise positive, *boundary defining function* $\rho_H \in \mathcal{C}^\infty(M)$ with differential non-zero on H; conversely H determines ρ_H up to a positive smooth multiple. The set of connected boundary hypersurfaces is denoted $\mathcal{M}_1(M)$ and the *boundary faces* of M are the *components* of the intersections of elements of $\mathcal{M}_1(M)$. We denote by $\mathcal{M}_k(M)$ the set of boundary faces of codimension k. Thus if $F \in \mathcal{M}_k(M)$ and $F' \in \mathcal{M}_{k'}(M)$ then $F \cap F'$ can be identified with the union over the elements of a subset (possibly empty of course) which we may denote $F \cap F' \subset \mathcal{M}_{k+k'}(M)$. Once again it is convenient to call a subset of $\mathcal{M}_k(M)$ with non-intersecting elements a collective boundary face, and then the collection of intersections of the elements of two collective boundary faces *is* a collective boundary face.

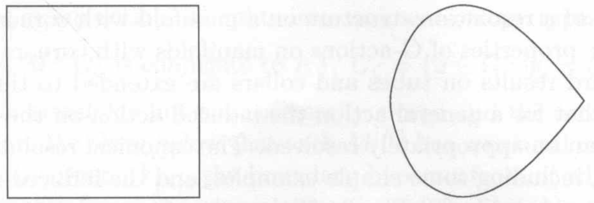

FIGURE 1. The square is a manifold with corners. The teardrop is only a tied manifold since its boundary hypersurface intersects itself.

By a manifold from now on we shall mean a manifold with corners, so the qualifier will be omitted except where emphasis seems appropriate. The traditional object will be called a boundaryless manifold.

As a consequence of the assumption that the boundary hypersurfaces are embedded, each boundary face of M is itself a manifold with corners (for a tied manifold the boundary hypersurfaces are more general objects, namely *articulated manifolds* which have boundary faces identified). At each point of a manifold with corners there are, by definition, *local product coordinates* $x_i \geq 0$, y_j where $1 \leq i \leq k$ and $1 \leq j \leq m-k$ (and either k or $m-k$ can be zero) and the x_i define the boundary hypersurfaces through the point. Unless otherwise stated, by local coordinates we mean local product coordinates in this sense. The local product structure near the boundary can be globalized:-

DEFINITION 1.1. On a compact manifold with corners, M, a *boundary product structure* consists of a choice $\rho_H \in \mathcal{C}^\infty(M)$ for each $H \in \mathcal{M}_1(M)$, of a defining function for the each of the boundary hypersurfaces, an open neighborhood $U_H \subset M$ of each $H \in \mathcal{M}_1(M)$ and a smooth vector field V_H defined in each U_H such that

$$V_H \rho_K = \begin{cases} 1 & \text{in } U_H \text{ if } K = H \\ 0 & \text{in } U_H \cap U_K \text{ if } K \neq H, \end{cases} \tag{1.3}$$
$$[V_H, V_K] = 0 \text{ in } U_H \cap U_K \ \forall \ H, K \in \mathcal{M}_1(M).$$

Integration of each V_H from H gives a product decomposition of a neighborhood of H as $[0, \epsilon_H] \times H$, $\epsilon_H > 0$ in which V_H is differentiation in the parameter space on which ρ_H induces the coordinate. Shrinking U_H allows it to be identified with such a neighborhood without changing the other properties (1.3). Scaling ρ_H and V_H allows the parameter range to be taken to be $[0, 1]$ for each H.

PROPOSITION 1.2. Every compact manifold has a boundary product structure.

PROOF. The construction of the neighborhoods U_H and normal vector fields V_H will be carried out inductively. For the inductive step it is convenient to consider a strengthened hypothesis. Note first that the data in (1.3) induces corresponding data on each boundary face F of M – where the hypersurfaces containing F are dropped, and for the remaining hypersurfaces the neighborhoods are intersected with F and the vector fields are restricted to F – to which they are necessarily tangent. It may be necessary to subdivide the neighborhoods if the intersection $F \cap H$ has more than one component. In particular this gives data as in (1.3) but with M replaced by F. So such data, with M replaced by one of its hypersurfaces, induces data on all boundary faces of that hypersurface. Data as in (1.3) on a collection of boundary hypersurfaces of a manifold M, with the defining functions ρ_H fixed, is said to be consistent if all restrictions to a given boundary face of M are the same.

Now, let $\mathcal{B} \subset \mathcal{M}_1(M)$ be a collection of boundary hypersurfaces of a manifold M, on which boundary defining functions ρ_H have been chosen for each $H \in \mathcal{M}_1(M)$, and suppose that neighborhoods U_K and vector fields V_K have been found satisfying (1.3) for all $K \in \mathcal{B}$. If $H \in \mathcal{M}_1(M) \setminus \mathcal{B}$ then we claim that there is a choice of V_H and U_H such that (1.3) holds for all boundary hypersurfaces in $\mathcal{B} \cup \{H\}$, with the neighborhoods possibly shrunk. To see this we again proceed inductively, by seeking V_H only on the elements of a subset $\mathcal{B}' \subset \mathcal{B}$ but consistent on all common boundary faces. The subset $\mathcal{B}'$ can always be increased, since the addition of another element of $\mathcal{B} \setminus \mathcal{B}'$ to $\mathcal{B}'$ requires the same inductive step but in lower overall dimension, which we can assume already proved. Thus we may assume that V_H has been constructed consistently on all elements of $\mathcal{B}$. Using the vector fields V_K, each of which is defined in the neighborhood U_K of K, V_H can be extended, locally uniquely, from the neighborhood of $K \cap H$ in K on which it is defined to a neighborhood of $K \cap H$ in M by demanding

$$\mathcal{L}_{V_K} V_H = [V_K, V_H] = 0. \tag{1.4}$$

The commutation condition and other identities follow from this and the fact that they hold on K. Moreover, the fact that the V_K commute in the intersections of the U_K means that these extensions of V_H are consistent for different K on their common domains. In this way V_H satisfying all conditions in (1.3) has been constructed in a neighborhood of the part of the boundary of H in M corresponding to $\mathcal{B}$. In the complement of this part of the boundary one can certainly choose V_H to satisfy $V_H \rho_H = 1$ and combining these two choices using a partition of unity (with two elements) gives the desired additional vector field V_H once the various neighborhoods U_K are shrunk.

Thus, after a finite number of steps the commuting normal vector fields V_K are constructed near each boundary hypersurface. □

Note that this result is equally true if in the definition the set of boundary hypersurfaces is replaced with any partition into collective boundary hypersurfaces, however it is crucial that the different hypersurfaces in each collection do not intersect.

The existence of such normal neighborhoods of the boundary hypersurfaces ensures the existence of 'product-type' metrics. That is, one can choose a metric g globally on M which near each boundary hypersurface H is of the form $d\rho_H^2 + \phi_H^* h_H$ where $\phi_H : U_H \longrightarrow H$ is the projection along the integral curves of V_H and h_H is a metric, inductively of the same product-type, on H. Thus near a boundary face $F \in \mathcal{M}_k(M)$, which is defined by ρ_{H_i}, $i = 1, \dots, k$, the metric takes the form

$$g = \sum_{i=1}^{k} d\rho_{H_i}^2 + \phi_F^* h_F \tag{1.5}$$

where ϕ_F is the local projection onto F with leaves the integral surfaces of the k commuting vector fields V_{H_i}. In particular

COROLLARY 1.3. On any manifold with corners there exists a metric g, smooth and non-degenerate up to all boundary faces, for which the boundary faces are each totally geodesic.

A diffeomorphism of a manifold sends boundary faces to boundary faces – which is to say there is an induced action on $\mathcal{M}_1(M)$.

DEFINITION 1.4. A diffeomorphism F of a manifold M is said to be *boundary intersection free* if for each $H \in \mathcal{M}_1(M)$ either $F(H) = H$ or $F(H) \cap H = \emptyset$. More generally a collection $\mathcal{G}$ of diffeomorphisms is said to be boundary intersection free if $\mathcal{M}_1(M)$ can be partitioned into collective boundary hypersurfaces $B_i \subset \mathcal{M}_1(M)$, so the elements of each B_i are disjoint, such that the induced action of each $F \in \mathcal{G}$ preserves the partition, i.e. maps each B_i to itself.

A manifold with corners, M, can always be realized as an embedded submanifold of a boundaryless manifold. As shown in **[12]**, if $\mathcal{F} \subset \mathcal{M}_1(M)$ is any disjoint collection of boundary hypersurfaces then the 'double' of M across $\mathcal{F}$, meaning $2_{\mathcal{F}}M = M \sqcup M/ \cup \mathcal{F}$ can be given (not however naturally) the structure of a smooth manifold with corners. If $\{\mathcal{F}_1, \dots \mathcal{F}_\ell\}$ is a partition of the boundary of M into disjoint collections, then it induces a partition $\{\widetilde{\mathcal{F}}_2, \dots \widetilde{\mathcal{F}}_\ell\}$ of the boundary of $2_{\mathcal{F}_1}M$ with one less element. After a finite number of steps, the iteratively doubled manifold is boundaryless and M may be identified with the image of one of the summands (see Theorem 4.2).

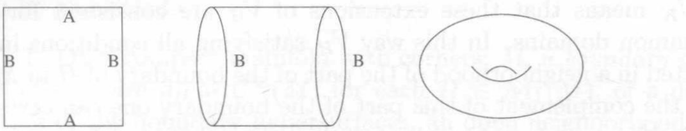

FIGURE 2. After doubling the boundaries marked A and then doubling the boundaries marked B we end up with a torus.

2. Blow up

A subset $X \subset M$ of a manifold (with corners) is said to be a *p-submanifold* if at each point of X there are local (product) coordinates for M such that $X \cap U$, where U is the coordinate neighborhood, is the common zero set of a subset of the coordinates. An *interior p-submanifold* is a p-submanifold no component of which is contained in the boundary of M.

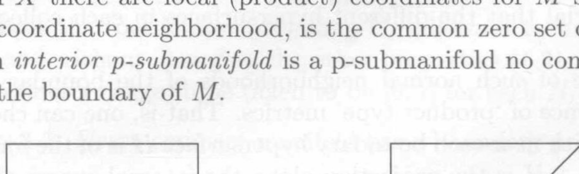

FIGURE 3. A horizontal line is an interior p-submanifold of the square. The diagonal in a product of manifolds with boundary is not a p-submanifold.

A p-submanifold of a manifold is itself a manifold with corners, and the collar neighborhood theorem holds in this context. Thus the normal bundle to X in M has (for a boundary p-submanifold) a well-defined inward-pointing subset, forming a submanifold with corners $N^+X \subset NX$ (defined by the non-negativity of all $d\rho_H$ which vanish on the submanifold near the point) and, as in the boundaryless case, the exponential map, but here for a product-type metric, gives a diffeomorphism of a neighborhood of the zero section with a neighborhood of X :

$$T : N^+X \supset U' \longrightarrow U \subset M. \tag{2.1}$$

The radial vector field on N^+X induces a vector field R near X which is tangent to all boundary faces.

PROPOSITION 2.1. *If X is a closed p-submanifold in a compact manifold then the boundary product structure in Proposition 1.2, for any choice of boundary defining functions, can be chosen so that V_H is tangent to X unless X is contained in H.*

PROOF. The condition that the V_H be tangent to X can be carried along in the inductive proof in Proposition 1.2, starting from the smallest boundary face which meets X. □

If $X \subset M$ is a closed p-submanifold then the radial blow-up of M along X is a well-defined manifold with corners $[M; X]$ obtained from M by replacing X by the inward-pointing part of its spherical normal bundle. It comes equipped with the blow-down map

$$[M; X] = S^+X \sqcup (M \setminus X),\ \beta : [M; X] \longrightarrow M. \tag{2.2}$$

The preimage of X, S^+X, is the 'front face' of the blow up, denoted $\mathrm{ff}([M; X])$. The natural smooth structure on $[M; X]$, with respect to which β is smooth, is characterized by the additional condition that a radial vector field R for X, as described above, lifts under β (i.e. is β-related) to $\rho_{\mathrm{ff}} X_{\mathrm{ff}}$ for a defining function ρ_{ff} and normal vector field X_{ff} for the new boundary introduced by the blow up.

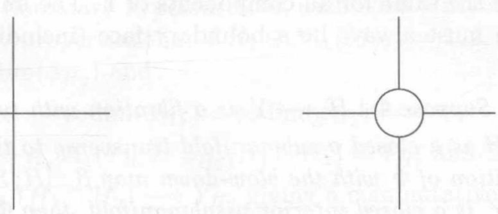

FIGURE 4. Blowing up the origin in $\mathbb{R}^2$ results in the manifold with boundary $[\mathbb{R}^2; \{0\}] = \mathbb{S}^1 \times \mathbb{R}^+$. Polar coordinates around the origin in $\mathbb{R}^2$ yield local coordinates near the front face in $[\mathbb{R}^2; \{0\}]$.

Except in the trivial cases that $X = M$ or $X \in \mathcal{M}_1(M)$ the front face is a 'new' boundary hypersurface of $[M; X]$ and the preimages of the boundary hypersurfaces of M are unions of the other boundary hypersurfaces of $[M; X]$; namely the lift of H is naturally $[H; X \cap H]$. So, in the non-trivial cases and unless X separates some boundary hypersurface into two components, there is a natural identification

$$\mathcal{M}_1([M; X]) = \mathcal{M}_1(M) \sqcup \{\mathrm{ff}([M; X])\} \tag{2.3}$$

which corresponds to each boundary hypersurface of M having a unique 'lift' to $[M;X]$, as the boundary hypersurface which is the closure of the preimage of its complement with respect to X. In local coordinates, blowing-up X corresponds to introducing polar coordinates around X in M.

LEMMA 2.2. *If X is a closed interior p-submanifold and M is equipped with a boundary product structure in the sense of Proposition 1.2 the normal vector fields of which are tangent to X then the radial vector field for X induced by the exponential map of an associated product-type metric commutes with V_H near any $H \in \mathcal{M}_1(M)$ which intersects X and on lifting to $[M;X]$, $R = \rho_{\mathrm{ff}} X_{\mathrm{ff}}$ where ρ_{ff} and X_{ff}, together with the lifts of the ρ_H and V_H give a boundary product structure on $[M;X]$.*

PROOF. After blow up of X the radial vector field lifts to be of the form $a\rho_{\mathrm{ff}}V_{\mathrm{ff}}$ for any normal vector field and defining function for the front face, with $a > 0$. The other product data lifts to product data for all the non-front faces of $[M;X]$ and this lifted data satisfies $[R, V_H] = 0$ near ff . Thus it is only necessary to show, using an inductive argument as above, that one can choose ρ_{ff} to satisfy $V_H \rho_{\mathrm{ff}} = 0$ and $R\rho_{\mathrm{ff}} = \rho_{\mathrm{ff}}$ in appropriate sets to conclude that $R = \rho_{\mathrm{ff}} V_{\mathrm{ff}}$ as desired. □

3. Resolution structures

A fibration is a surjective smooth map $\Phi : H \longrightarrow Y$ between manifolds with the property that for each component of Y there is a manifold Z such that each point p in that component has a neighborhood U for which there is a diffeomorphism giving a commutative diagram with the projection onto U :

$$\begin{array}{ccc} \Phi^{-1}(U) & \xrightarrow{\;F_U\;} & Z \times U \\ & {\scriptstyle\Phi}\searrow \quad \swarrow{\scriptstyle\pi_U} & \\ & U. & \end{array} \tag{3.1}$$

The pair (U, F_U) is a local trivialization of Φ. Set $\operatorname{codim}(\phi) = \dim Z$, which will be assumed to be the same for all components of Y. The image of a boundary face under a fibration must always be a boundary face (including the possibility of a component of Y).

LEMMA 3.1. *Suppose $\Phi : H \longrightarrow Y$ is a fibration with typical fiber Z.*

i) *If $S \subseteq H$ is a closed p-submanifold transverse to the fibers of Φ, then the composition of Φ with the blow-down map $\beta : [H;S] \longrightarrow H$ is a fibration.*
ii) *If $T \subseteq Y$ is a closed interior p-submanifold, then Φ lifts from $H \setminus \Phi^{-1}(T)$ to a fibration $\beta^{\#}\Phi : [H; \Phi^{-1}(T)] \longrightarrow [Y;T]$.*

REMARK 3.2. In the situation of ii), one may consider instead the pull-back fibration

$$\begin{array}{ccc} \beta_Y^* H & \longrightarrow & H \\ \downarrow & & \downarrow{\scriptstyle\Phi} \\ [Y;T] & \xrightarrow{\;\beta_Y\;} & Y \end{array}$$

where $\beta_Y^* H = \{(\zeta, \xi) \in H \times [Y;T] : \Phi(\zeta) = \beta_Y(\xi)\}$. The natural map $[H; \Phi^{-1}(T)] \ni \alpha \mapsto (\beta_H(\alpha), \widetilde{\Phi}(\alpha)) \in \beta_Y^* H$ is a diffeomorphism, showing that these fibrations coincide.

PROOF. i) Transversality ensures that $\Phi(S) = Y$ and so $\Phi\big|_S$ is itself a fibration, say with typical fiber Z_S. If (U, F_U) is a local trivialization of Φ then since

$$[U \times Z; U \times Z_S] = U \times [Z; Z_S],$$

the diffeomorphism F_U induces a diagram

$$\begin{array}{ccc} (\beta^*\Phi)^{-1}(U) & \longrightarrow & U \times [Z; Z_S] \\ & {\scriptstyle \beta^*\Phi} \searrow \quad \swarrow {\scriptstyle \pi_U} & \\ & U & \end{array}$$

which shows that $\beta^*\Phi : [H; S] \longrightarrow H \longrightarrow Y$ is a fibration.
ii) Let (U, F_U) be a local trivialization of Φ and $T_U = T \cap U$. The diffeomorphism F_U identifies $\Phi^{-1}(U)$ with $Z \times U$ and $\Phi^{-1}(T_U)$ with $Z \times T_U$ and so lifts to a diffeomorphism $\widetilde{F}_U$ of $(\beta^\#\Phi)^{-1}([U; T_U])$ with $Z \times [U; T_U] = [Z \times U; Z \times T_U]$. Thus $([U; T_U], \widetilde{F}_U)$ is a local trivialization for $\beta^\#\Phi$,

$$\begin{array}{ccc} (\beta^\#\Phi)^{-1}([U; T_U]) & \xrightarrow{\widetilde{F}_U} & Z \times [U; T_U] \\ & {\scriptstyle \beta^\#\Phi} \searrow \quad \swarrow {\scriptstyle \pi_U} & \\ & [U; T_U] & \end{array}$$

which shows that $\beta^\#\Phi : [H; \Phi^{-1}(T)] \longrightarrow [Y; T]$ is a fibration. □

The restriction of the blow-down map to the boundary hypersurface introduced by the blow up of a p-submanifold is a fibration, just the bundle projection for the (inward-pointing part of) the normal sphere bundle. In general repeated blow up will destroy the fibration property of this map. However in the resolution of a G-action the fibration condition persists. We put this into a slightly abstract setting as follows.

DEFINITION 3.3. A resolution structure on a manifold M is a partition of $\mathcal{M}_1(M)$ into collective boundary hypersurfaces, each with a fibration, $\phi_H : H \longrightarrow Y_H$ with the consistency properties that if $H_i \in \mathcal{M}_1(M)$, $i = 1, 2$, and $H_1 \cap H_2 \neq \emptyset$ then $\operatorname{codim}(\phi_{H_1}) \neq \operatorname{codim}(\phi_{H_2})$ and

$$\operatorname{codim}(\phi_{H_1}) < \operatorname{codim}(\phi_{H_2}) \Longrightarrow$$
$$\phi_{H_1}(H_1 \cap H_2) \in \mathcal{M}_1(Y_{H_1}),\ \phi_{H_2}(H_1 \cap H_2) = Y_{H_2} \text{ and } \exists \text{ a fibration}$$
$$\phi_{H_1 H_2} : \phi_{H_1}(H_1 \cap H_2) \longrightarrow Y_{H_2} \text{ giving a commutative diagram:}$$

(3.2)
$$\begin{array}{ccc} H_1 \cap H_2 & \xrightarrow{\phi_{H_1}} & \phi_{H_1}(H_1 \cap H_2) \\ & {\scriptstyle \phi_{H_2}} \searrow \quad \swarrow {\scriptstyle \phi_{H_1 H_2}} & \\ & Y_{H_2}. & \end{array}$$

LEMMA 3.4. *A resolution structure induces resolution structures on each of the manifolds* Y_H.

PROOF. Each boundary hypersurface F of Y_H is necessarily the image under ϕ_H of a unique boundary hypersurface of H, therefore consisting of a component of some intersection $H \cap K$ for $K \in \mathcal{M}_1(M)$. The condition (3.2) ensures that

$\operatorname{codim}(\phi_H) < \operatorname{codim}(\phi_K)$ and gives the fibration $\phi_{HK} : F \longrightarrow Y_K$. Thus for Y_H the bases of the fibrations of its boundary hypersurfaces are all the Y_K's with the property that $H \cap K \neq \emptyset$ and $\operatorname{codim}(\phi_H) < \operatorname{codim}(\phi_K)$ with the fibrations being the appropriate maps ϕ_* from (3.2).

Similarly the compatibility maps for the boundary fibration of Y_H follow by the analysis of the intersection of three boundary hypersurfaces H, K and J where $\operatorname{codim}(\phi_H) < \operatorname{codim}(\phi_K) < \operatorname{codim}(\phi_J)$. Any two intersecting boundary hypersurfaces of Y_H must arise in this way, as $\phi_H(H \cap K)$ and $\phi_H(H \cap J)$ and the compatibility map for them is ϕ_{JK}. □

If M carries a resolution structure then Lemma 3.1 shows that appropriately placed submanifolds can be blown up and the resolution structure can be lifted. Specifically we say that a manifold T is *transverse to the resolution structure* if either:

i) T is an interior p-submanifold of M, with $\dim T < \dim M$, that is transverse to the fibers of ϕ_H for all $H \in \mathcal{M}_1(M)$, or

ii) T is an interior p-submanifold of Y_L, for some $L \in \mathcal{M}_1(M)$, with $\dim T < \dim Y_L$, that is transverse to the fibers of ϕ_N for all $N \in \mathcal{M}_1(Y_L)$.

Let $\widetilde{T} \subseteq M$ be equal to T in the first case and $\phi_L^{-1}(T)$ in the second, then we have the following result.

PROPOSITION 3.5. *If M carries a resolution structure and T is a manifold transverse to it, then $[M; \widetilde{T}]$ carries a resolution structure. In case ii) above, where $T \subseteq Y_L$, the resolution structure on $[M; \phi_L^{-1}(T)]$ is obtained by blowing-up the lift of T to every Y_K that fibers over Y_L. In both cases, at each boundary face of the new resolution structure the boundary fibration is either the pull-back of the previous one along the blow-down map or the blow-down map itself.*

Recall that submanifolds which do not intersect are included in the notion of transversal intersection.

PROOF. Consider the two cases in the definition of transverse submanifold separately. (For clarity, we assume throughout the proof that the collective boundary hypersurfaces in Definition 3.3 consist of a single boundary hypersurface.)
Case i). Let $\beta_T : [M; T] \longrightarrow M$ be the blow-down map. A boundary face of $[M; T]$ is either the lift of a boundary face $H \in \mathcal{M}_1(M)$, in which case $\beta_T^* \phi_H$ is a fibration by Lemma 3.1 i), or it is the front face of the blow-up, in which case it carries the fibration $\beta_T\big|_{\mathrm{ff}}$. Thus we only need to check the compatibility conditions.

The compatibility maps for the fibrations of the hypersurfaces of M clearly lift to give compatibility maps for the lifts. Thus it is only necessary to check compatibility between the fibrations on these lifted boundary hypersurfaces of $[M; T]$ and that of the front face. So, let H be a hypersurface of M that intersects T. In terms of the notation above, the codimension of $\beta_T^* \phi_H$ is the equal to $\dim Z_H$ while the codimension of ϕ_{ff} is equal to $\dim Z_H - \dim Z_{H \cap T}$. The diagram (3.2) in this case is

$$\begin{array}{ccc} \mathrm{ff} \cap [H; H \cap T] & \xrightarrow{\quad \beta_T \quad} & H \cap T \\ & {\scriptstyle \widetilde{\phi}_H} \searrow \quad \swarrow {\scriptstyle \phi_H} & \\ & \phi_H(H \cap T) = Y_H. & \end{array}$$

and so the requirements of Definition 3.3 are met.
Case ii). First note that the inverse image of a p-submanifold under a fibration is again a p-submanifold since this is a local property and locally a fibration is a projection. We denote by $\beta_T : [M; \phi_L^{-1}(T)] \longrightarrow M$ the blow-down map and make use of the notation in (3.2).

From the front face the map

$$\mathrm{ff}([M;\phi_L^{-1}(T)]) \xrightarrow{\beta_T} \phi_L^{-1}(T) \xrightarrow{\phi_L} T$$

is the composition of fibrations and so is itself a fibration.

Consider the lift of a boundary face $H \in \mathcal{M}_1(M)$ to a boundary face of $[M;\widetilde{T}]$. If $H \cap \phi_L^{-1}(T)$ is empty then $\beta_T^*\phi_H$ fibers over Y_H and the compatibility conditions are immediate. If $H \cap \phi_L^{-1}(T)$ is not empty and $\operatorname{codim}(\phi_L) < \operatorname{codim}(\phi_H)$ then, by Lemma 3.1, $\beta_T^*\phi_H$ fibers over Y_H and the arrows in the commutative diagrams

$$\begin{array}{ccc} [H\cap L; H\cap\phi_L^{-1}(T)] & \xrightarrow{\beta_T^{\#}\phi_L} & [\phi_L(H\cap L);\phi_L(H\cap L)\cap T] \\ & {\scriptstyle \beta_T^*\phi_H}\searrow \qquad \swarrow{\scriptstyle \beta^*\phi_{LH}} & \\ & Y_H & \end{array}$$

and

$$\begin{array}{ccc} \mathrm{ff}([H; H\cap\phi_L^{-1}(T)]) & \xrightarrow{\beta_T^*\phi_L} & \phi_L(H\cap L)\cap T \\ & {\scriptstyle \beta_T^*\phi_H}\searrow \qquad \swarrow{\scriptstyle \phi_{LH}} & \\ & Y_H & \end{array}$$

are all fibrations. Here, surjectivity of $\phi_{LH}\big|_{\phi_L(H\cap L)\cap T}$ follows from the transversality of T to the fibers of ϕ_{LH}. Since the lift of H meets the lift of L in $[H\cap L; H\cap\phi_L^{-1}(T)]$ and meets the front face of $[M;\phi_L^{-1}(T)]$ in $\mathrm{ff}([H;\phi_L^{-1}(T)\cap H])$, these diagrams also establish the compatibility conditions for the lift of H.

Next if $H\cap\phi_L^{-1}(T)$ is not empty and $\operatorname{codim}(\phi_L) > \operatorname{codim}(\phi_H)$, then Lemma 3.1 guarantees that the map $\beta_T^{\#}\phi_H$ is a fibration from the lift of H to $[Y_H;\phi_{HL}^{-1}(T)]$ and that the arrows in the commutative diagrams

$$\begin{array}{ccc} [H\cap L; H\cap\phi_L^{-1}(T)] & \xrightarrow{\beta_T^{\#}\phi_H} & [\phi_L(H\cap L);\phi_L(H\cap L)\cap T] \\ & {\scriptstyle \beta_T^{\#}\phi_H}\searrow \qquad \swarrow{\scriptstyle \beta^{\#}\phi_{HL}} & \\ & [Y_L;T] & \end{array}$$

and

$$\begin{array}{ccc} \mathrm{ff}([H; H\cap\phi_L^{-1}(T)]) & \xrightarrow{\beta_T^{\#}\phi_L} & \mathrm{ff}([\phi_H(H\cap L);\phi_{HL}^{-1}(T)]) \\ & {\scriptstyle \beta_T^*\phi_H}\searrow \qquad \swarrow{\scriptstyle \beta^*\phi_{HL}} & \\ & T. & \end{array}$$

are all fibrations.

Finally consider the lift of L. The map $\beta^{\#}\phi_L : [L;\phi_L^{-1}(T)] \longrightarrow [Y_L;T]$ is a fibration by Lemma 3.1 and the discussion above shows that it is compatible with