Ergodic Theory, Analysis, and Efficient Simulation of Dynamical Systems

Bernold Fiedler
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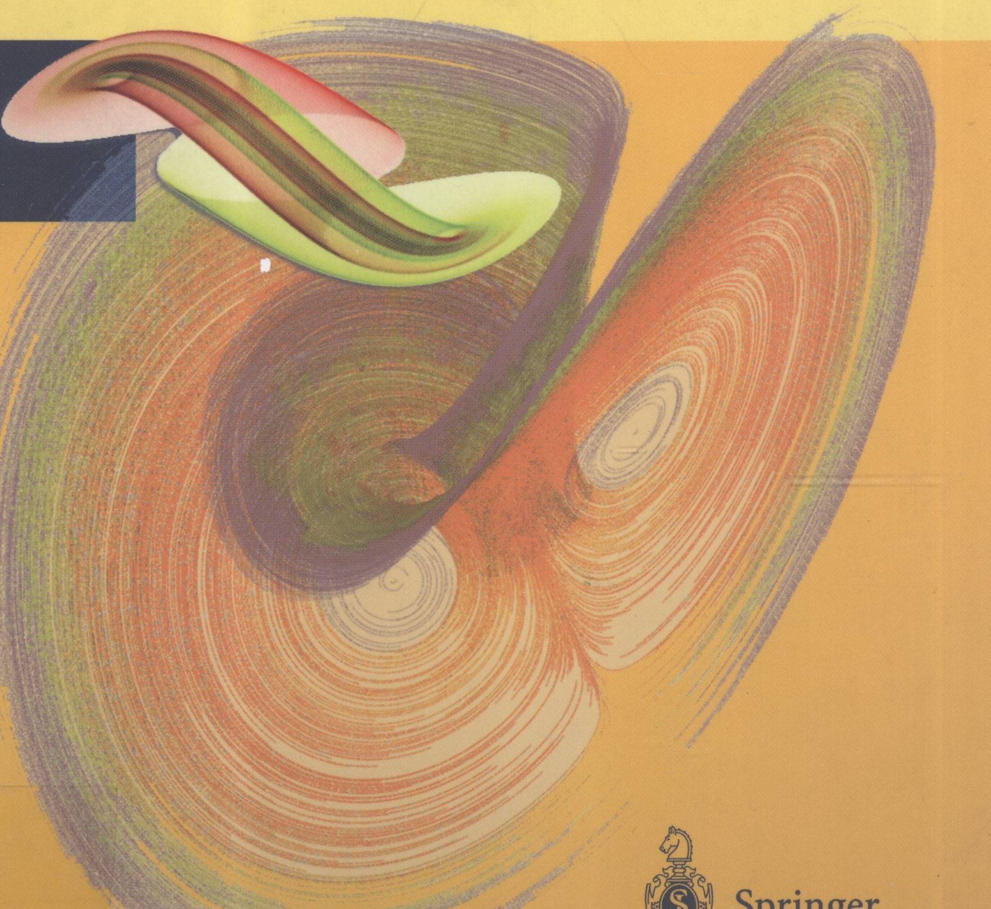

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Library of Congress Cataloging-in-Publication Data

Ergodic theory, analysis, and efficient simulation of dynamical systems / Bernold Fiedler (editor).
p. cm.
Includes bibliographical references and index.
ISBN 3540412905 (acid-free paper)
1. Differentiable dynamical systems. 2. Ergodic theory. 3. Mathematical analysis. I. Fiedler, Bernold, 1956-

QA614.8 .E737 2001
515'.35--dc21

2001020854

Mathematics Subject Classification (2000): 28, 34, 35, 65, 81, 92

ISBN 3-540-41290-5 Springer-Verlag Berlin Heidelberg New York

Springer-Verlag Berlin Heidelberg New York
a member of BertelsmannSpringer Science+Business Media GmbH
http://www.springer.de

Printed in Germany

Cover design: *Erich Kirchner, Heidelberg*
Typsetting: Le-TeX Jelonek, Schmidt & Vöckler GbR, Leipzig
Printed on acid-free paper SPIN 10789119 46/3142ck-5 4 3 2 1 0

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and Efficient Simulation of Dynamical Systems

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Preface

This book summarizes and highlights progress in our understanding of Dynamical Systems during six years of the German Priority Research Program "Ergodic Theory, Analysis, and Efficient Simulation of Dynamical Systems". The program was funded by the Deutsche Forschungsgemeinschaft (DFG) and aimed at combining, focussing, and enhancing research efforts of active groups in the field by cooperation on a federal level. The surveys in the book are addressed to experts and non-experts in the mathematical community alike. In addition they intend to convey the significance of the results for applications far into the neighboring disciplines of Science.

Three fundamental topics in Dynamical Systems are at the core of our research effort:

- behavior for large time
- dimension
- measure, and chaos

Each of these topics is, of course, a highly complex problem area in itself and does not fit naturally into the deplorably traditional confines of any of the disciplines of ergodic theory, analysis, or numerical analysis alone. The necessity of mathematical cooperation between these three disciplines is quite obvious when facing the formidable task of establishing a bidirectional transfer which bridges the gap between deep, detailed theoretical insight and relevant, specific applications. Both analysis and numerical analysis play a key role when it comes to building that bridge. Some steps of our joint bridging efforts are collected in this volume.

Neither our approach nor the presentations in this volume are monolithic. Rather, like composite materials, the contributions are gaining strength and versatility through the broad variety of interwoven concepts and mathematical methodologies which they span.

Fundamental concepts which are present in this volume include bifurcation, homoclinicity, invariant sets and attractors, both in the autonomous and nonautonomous situation. These concepts, at first sight, seem to mostly address *large time behavior*, most amenable to methodologies of analysis. Their intimate relation to concepts like (nonstrict) hyperbolicity, ergodicity, entropy, stochasticity and control should become quite apparent, however, when browsing through this volume.

The fundamental topic of *dimension* is similarly ubiquitous throughout our articles. In analysis it figures, for example, as a rigorous reduction from infinite-dimensional settings like partial differential equations, to simpler infinite-, finite- or even low-dimensional model equations, still bearing full relevance to the original equations. But in numerical analysis – including and transcending mere discretization – specific computational realization of such reductions still poses challenges which are addressed here.

Another source of inspiration comes from very refined *measure*-theoretic and dimensional concepts of ergodic theory which found their way into algorithmic realizations presented here.

By no means do these few hints exhaust the conceptual span of the articles. It would be even more demanding to discuss the rich circle of methods, by which the three fundamental topics of large time behavior, dimension, and measure are tackled. In addition to SBR-measures, Perron-Frobenius type transfer operators, Markov decompositions, Pesin theory, entropy, and Oseledets theorems, we address kneading invariants, fractal geometry and self-similarity, complex analytic structure, the links between billiards and spectral theory, Lyapunov exponents, and dimension estimates. Including Lyapunov-Schmidt and center manifold reductions together with their Shilnikov and Lin variants and their efficient numerical realizations, symmetry and orbit space reductions together with closely related averaging methods, we may continue, numerically, with invariant subspaces, Godunov type discretization schemes for conservation laws with source terms, (compressed) visualization of complicated and complex patterns of dynamics, and present an algorithm, GAIO, which enables us to approximately compute, in low dimensions, objects like SBR-measures and Perron-Frobenius type transfer operators. At which point our cursory excursion through methodologies employed here closes up the circle.

So much for the mathematical aspects. The range of applied issues, mostly from physics but including some topics from the life sciences, can also be summarized at most superficially, at this point. This range comprises such diverse areas as crystallization and dendrite growth, the dynamo effect, and efficient simulation of biomolecules. Fluid dynamics and reacting flows are addressed, including the much studied contexts of Rayleigh-Bénard and Taylor-Couette systems as well as the stability question of three-dimensional surface waves. The Ginzburg-Landau and Swift-Hohenberg equations appear, for example, as do mechanical problems involving friction, population biology, the spread of infectious diseases, and quantum chaos. It is the diversity of these applied fields which well reflects both the diversity and the power of the underlying mathematical approach. Only composite materials enable a bridge to span that far.

The broad scope of our program has manifested itself in many meetings, conferences, and workshops. Suffice it to mention the workshop on "Entropy" which was coorganized by Andreas Greven, Gerhard Keller, and Gerald Warnecke at Dresden in June 2000, jointly with the two neighboring DFG Priority Research Programs "Analysis and Numerics for Conservation Laws" and "Interacting Stochastic Systems of High Complexity". For further information concerning program and participants of the DFG Priority Research Program "Ergodic Theory, Analysis, and Efficient Simulation of Dynamical Systems", including a preprint server, see

- www.math.fu-berlin.de/~ danse/

For other DFG programs we refer to

- www.dfg.de
- www.dfg.de/aufgaben/Schwerpunktprogramme.html

At the end of this preface, I would like to thank at least some of the many friends and colleagues who have helped on so many occasions to make this program work. First of all, I would like to mention the members of the scientific committee who have helped initiate the entire program and who have accompanied and shaped the scientific program throughout its funding period: Ludwig Arnold, Hans-Günther Bothe, Peter Deuflhard, Klaus Kirchgässner, and Stefan Müller. The precarious conflict between great expectations and finite funding was expertly balanced by our all-understanding referees Hans Wilhelm Alt, Jürgen Gärtner, François Ledrappier, Wilhelm Niethammer, Albrecht Pietsch, Gerhard Wanner, Harry Yserentant, Eberhard Zeidler, and Eduard Zehnder. The hardships of finite funding as well as any remaining administrative constraints were further alleviated as much as possible, and beyond, by Robert Paul Königs and Bernhard Nunner, representing DFG at its best. The www-services were designed, constantly expanded and improved with unrivalled expertise and independence by Stefan Liebscher. And Regina Löhr, as an aside to her numerous other secretarial activities and with ever-lasting patience and friendliness, efficiently reduced the administrative burden of the coordinator to occasional emails which consisted of no more than "OK. BF". Martin Peters and his team at Springer-Verlag ensured a very smooth cooperation, including efficient assistance with all TEXnicalities. But last, and above all, my thanks as a coordinator of this program go to the authors of this volume and to all participants – principal investigators, PostDocs and students alike – who have realized this program with their contributions, their knowledge, their dedication, and their imagination.

Berlin,
September 2000

Bernold Fiedler

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Random Attractors: Robustness, Numerics and Chaotic Dynamics

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Abstract. In this article the numerical approximation of attractors and invariant measures for random dynamical systems by using a box covering algorithm is discussed. We give a condition under which the algorithm, which defines a set valued random dynamical system, possesses an attractor close to the attractor of the original system. Furthermore, a general existence theorem for attractors for set valued random dynamical systems is proved and criteria for the robustness of random attractors under perturbations of the system are given. Our numerical algorithm is applied to the stochastically forced Duffing oscillator, which supports for certain parameter values a non–trivial random SRB measure.

1 Introduction

The qualitative behavior of an autonomous dynamical system given by the iterations of a map or the solution of an ordinary differential equation (ODE) is often characterized by invariant objects such as (global) attractors or invariant (probability) measures. Invariant measures describe the statistical behavior of a dynamical system. An attractor is a compact invariant set, which carries the asymptotic dynamics in the sense that it is approached by all trajectories under time evolution. If a global attractor exists, then it supports all invariant measures.

These "global objects" are particularly valuable in the investigation of systems with complicated ("chaotic") dynamical behavior, where reliable predictions for single trajectories are only possible for bounded time intervals.

Here we are interested in dynamics influenced by probabilistic noise. A "traditional" approach is to model this by a Markov process. In this case the evolution law of the system is given by transition probabilities instead of a deterministic map. The statistical behavior of a Markov process is often described by probability measures, which are invariant under the transition probabilities.

However, we will work in the framework of random dynamical systems (for a systematic presentation of this theory see [1]). A discrete time random dynamical system is just given by the iteration of random mappings. Stochastic differential equations (SDE's) and random differential equations (which

* Project: The Multiplicative Ergodic Theorem under Discretization and Perturbation (Ludwig Arnold and Wolfgang Krieger)

are ODE's with a randomly varying parameter) are generators of continuous time random dynamical systems.

Iterations of iid mappings and SDE's also define Markov processes. There is a notion of invariant measures for random dynamical systems, which are closely related to the invariant measures of the corresponding Markov process. However, typically a random dynamical system has more invariant measures than just these "Markov measures". The theory of random dynamical systems provides tools which allow a more detailed analysis of the dynamics than it is possible in the framework of Markov processes. In some sense it is possible to separate "noise driven dynamics" from "deterministic" dynamics.

There is a notion of a "pathwise" defined random attractor. That is, the attractor is a (compact) set valued random variable defined on the probability space which models the noise, i.e. for (almost) every realization of the stochastic process which models the noise there is a compact set. The distance between a trajectory with random initial condition and the attractor converges to zero in probability if time tends to infinity.

Such an attractor moves (in a stationary manner) in the phase space of the system under time evolution. This movement of the attractor can be interpreted as the "noise induced part" of the dynamics. In addition, there is some dynamics inside the attractor (which may be trivial). The qualitative structure of this dynamics inside the attractor is often the same for (almost) every noise realization. In this sense it can be viewed as the "deterministic part" of the dynamics.

To be more precise, let us look at two examples. Take some self mappings of a complete metric space, which are all uniform contractions. In every time step choose randomly one of these mappings. Then the asymptotic behavior of a trajectory is independent of the initial condition. This means, that the "deterministic part" of the dynamics is trivial. The attractor consists of one random point. However, if the mappings do not have a common fixed point, then an observer sees non–trivial dynamical behavior, which is in this case exclusively due to the noise.

The situation is different for the randomly forced Duffing oscillator, which will be discussed in more detail later on in this paper. There are parameter values, where the system has a positive Lyapunov exponent, which leads to sensitive dependence on initial conditions. Our numerical images of the attractor and of the "natural" measure supported by it suggest that the dynamics of the system is a combination of "deterministic dynamics" which is chaotic and "noise induced" movements in the phase space $\mathbb{R}^2$.

For many random dynamical systems the existence of a random attractor, which also implies the existence of invariant measures, is proved. However, in many interesting cases it seems to be quite hard to obtain analytical results on the structure of the attractor and the dynamics on it (the "deterministic part"). This means that one often has to rely on numerical simulations in order to obtain information about random attractors and invariant measures.

In this article we will discuss a numerical method for the approximation of random attractors and ("natural") invariant measures supported by them. We consider a "global" approach in order to approximate these global objects. We use a random version of the subdivision algorithm which was developed by Dellnitz and Hohmann [14,15] for the approximation of deterministic attractors. This algorithm produces a sequence of box coverings of the attractor.

The "random" subdivision algorithm is described in an article by Keller and Ochs [25]. Its application to the stochastically perturbed Duffing–van der Pol oscillator gave surprising new insight in the structure of the attractor.

A simplifying statement about the present paper would be that its content is essentially a new convergence proof for this algorithm. However, on the way to this proof we develop some tools which may be of interest for their own. After giving fundamental definitions (Sect. 2) we introduce in Sect. 3 the notion of a set valued random dynamical system, where the image of a point is a compact set instead of a single point. We define attractors for set valued random dynamical systems and give a general criterion for their existence based on the existence of an attracting set. This generalizes a often used existence theorem for an attractor of a standard point valued random dynamical system.

Using the notion of set valued random dynamical systems we give criteria for robustness of random attractors under perturbations. In Sect. 4 we apply these results to the "random" subdivision algorithm, which defines for a given random dynamical system φ a sequence of set valued random dynamical systems $\hat{\varphi}_k$ each of which possessing an attractor A_k (under an assumption on φ, see Corollary 26). We show that the intersection of all A_k is the global attractor for φ.

In addition (Sect. 4.2) we discuss a method for the numerical approximation of "natural" invariant measures of random dynamical systems. In the case of random dynamical systems generated by stochastic differential equations these "natural" measures are related to invariant measures of the corresponding Markov process.

In Sect. 5 we consider as an example the stochastically forced damped Duffing oscillator. We prove the existence of a global attractor. There are parameters where the corresponding random dynamical system has a positive Lyapunov exponent. This implies that the "natural" measure is a non–trivial random Sinai–Ruelle–Bowen measure. We have calculated numerical approximations of the attractor and of this measure. In addition we consider the case when the top Lyapunov exponent is negative. Then the support of the "natural" measure is a random one point set, but the global attractor seems to be a larger set carrying some sort of "transient" chaotic dynamics.

2 Preliminaries

2.1 Random Dynamical Systems

Definition 1. A (continuous) *random dynamical system* (RDS) consists of two ingredients:

- A measure preserving flow $\vartheta = (\vartheta_t)_{t \in \mathbb{T}}$ on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, which serves as a model for the noise. We always assume that ϑ is invertible, i.e. $\mathbb{T} = \mathbb{R}$ or $\mathbb{Z}$.
- A measurable mapping

$$\varphi : \mathbb{T}^{(+)} \times \Omega \times X \to X, \quad (t, \omega, x) \mapsto \varphi(t, \omega)x$$

(with $\mathbb{T}^+ = \{t \in \mathbb{T} : t \geq 0\}$), where the *state space* X is a separable metric space (with metric d), such that
  - $(t, x) \mapsto \varphi(t, \omega)x$ is continuous for fixed ω,
  - φ satisfies the *cocycle property* $\varphi(0, \omega) = \mathrm{id}_X$ and $\varphi(t + s, \omega) = \varphi(s, \vartheta_t\omega) \circ \varphi(t, \omega)$ for $t, s \in \mathbb{T}^{(+)}$ and $\omega \in \Omega$.

We say φ is a random dynamical system on X over the *metric dynamical system* (Ω, ϑ).

Remark 2. (i) The assumption that X is separable is for technical reasons.

(ii) In some definitions continuity is only required in space (i.e. $x \mapsto \varphi(t, \omega)x$ is continuous for fixed t, ω). However, continuity in time is automatically true if $\mathbb{T} = \mathbb{Z}$ and is also satisfied for continuous time random dynamical systems generated by stochastic or random differential equations (see Arnold [1, Chapter 2]).

(iii) Throughout this paper all assertions about ω are assumed to hold on a ϑ invariant set of full measure (unless otherwise stated).

(iv) The cocycle property (which reduces to the flow property if φ is independent of the noise ω) implies, that the *skew product*

$$\Theta = \Theta_\varphi : \mathbb{T}^{(+)} \times \Omega \times X \to \Omega \times X, \quad (t, \omega, x) \mapsto \Theta(t)(\omega, x) := (\vartheta_t\omega, \varphi(t, \omega)x)$$

defines a measurable (semi-)flow on the product space $\Omega \times X$.

2.2 Generators

Iteration of Random Mappings. Let X be a metric space and $\mathbb{P}_0$ a probability measure on the space $C(X, X)$ of continuous mappings from X to itself. Set $\Omega := C(X, X)^{\mathbb{Z}}$ and let ϑ be the left shift on Ω. Define $\mathbb{P} := \mathbb{P}_0^{\mathbb{Z}}$. With $\vartheta_n = \vartheta^n$, $\varphi(\omega) = \omega_0$ and

$$\varphi(n, \omega) = \varphi(\vartheta_{n-1}\omega) \circ \dots \circ \varphi(\omega)$$

for $n \geq 1$ a random dynamical system on X over (Ω, ϑ) is defined, which models the iteration of iid mappings with distribution $\mathbb{P}_0$. More generally, a random dynamical system is defined with every ϑ invariant probability measure on Ω.

If $\mathbb{P}$ is concentrated on homeomorphisms, then $\varphi(n, \omega)$ is defined for $n < 0$ via

$$\varphi(n, \omega) = \varphi(\vartheta_{-n}\omega)^{-1} \circ \ldots \circ \varphi(\vartheta_{-1}\omega)^{-1}.$$

In the same way a discrete time random dynamical system is generated if we have an arbitrary metric dynamical system (Ω, ϑ) and a measurable mapping $\varphi : \Omega \to C(X, X)$.

Stochastic Differential Equations. Consider the Stratonovich SDE

$$dx = f(x)\, dt + \sum_{i=1}^{m} g_i(x) \circ dW_i$$

on $\mathbb{R}^d$ with initial condition $x_0 \in \mathbb{R}^d$ and functions $f, g_i : \mathbb{R}^d \to \mathbb{R}^d$ for $i = 1, \ldots, m$ and $W = (W_1, \ldots, W_m)$ an m-dimensional Wiener process. If the functions f, g_i are sufficiently regular (see Theorem 2.3.36 in Arnold [1]) this equation generates a local RDS φ over (Ω, ϑ), where $\Omega = C_0(\mathbb{R}, \mathbb{R}^m)$ is the space of continuous functions ω from $\mathbb{R}$ to $\mathbb{R}^m$ with $\omega(0) = 0$ (the path space of the Wiener process) equipped with the canonical Wiener measure, and the shift ϑ is defined by $(\vartheta_t\omega)(s) = \omega(s + t) - \omega(t)$ for all $s, t \in \mathbb{R}$ and $\omega \in \Omega$. Local means that $\varphi(t, \omega)\, x$ is only defined for $\tau^-(\omega, x) < t < \tau^+(\omega, x)$, where $-\tau^-, \tau^+ \in (0, \infty]$ are the lifetimes of solutions before possible explosion. A sufficient condition for φ being a global RDS without explosion is global Lipschitz continuity of the functions f, g_i (see Arnold [1, Theorem 2.3.32]).

Random Differential Equations. Here the generator of a continuous time random dynamical system φ is a family of ODE's with parameter ω, which can be solved "pathwise" for each ω (in contrast to the SDE case) as a deterministic non–autonomous ODE. That is, φ satisfies the integral equation

$$\varphi(t, \omega)x = x + \int_0^t f(\vartheta_s\omega, \varphi(s, \omega)x) ds,$$

where (Ω, ϑ) is any continuous time metric dynamical system and f is a function from $\Omega \times \mathbb{R}^d$ to $\mathbb{R}^d$.

We say that φ solves the *random differential equation* $\dot{x}(t) = f(\vartheta_t\omega, x(t))$. This pathwise differential equation is fulfilled if $(t, x) \mapsto f(\vartheta_t\omega, x)$ is continuous and $x \mapsto f(\omega, x)$ is Lipschitz. In this case we speak of a *classical solution.* Otherwise the differential equation is a symbolic notation for the corresponding integral equation. For general conditions on f which are needed to generate a random dynamical system see [1, Theorems 2.2.1 and 2.2.2].

2.3 Markov Processes

A (homogeneous) Markov process (see e.g. [1, Appendix A.4]) on X is given by an initial probability distribution at time 0 and a family $(P(t,x,\cdot))_{t\in\mathbb{T}^+,x\in X}$ of *transition probabilities*, i.e. $P(t,x,\cdot)$ is a probability measure on X for every $t\geq 0$, $x\in X$. For a Borel set $A\subset X$ the value $P(x,t,A)$ is the probability to end up in A at time $s+t$ if one starts in the point x at time s. The "flow property" of a Markov process is the *Chapman–Kolmogorov equation*

$$P(s+t,x,A)=\int_X P(s,y,A)P(t,x,dy).$$

If $P(t,x,\cdot)$ is a point mass in $\varphi_t x$, then a Markov process reduces to a semi-flow.

An *invariant (or stationary) measure* is a Borel probability measure ρ on X with

$$\rho(A)=\int_X P(t,x,A)d\rho(x).$$

A random dynamical system φ has *independent increments*, if $\varphi(t,\cdot)$ and $\varphi(u,\vartheta_s\cdot)$ are stochastically independent if $0\leq t\leq s$ and $u\geq 0$. This is the case if φ describes in discrete time the iteration of iid mappings or if φ is generated by an SDE.

Every random dynamical system φ with independent increments defines a Markov process with transition probabilities

$$P(t,x,A)=\mathbb{P}(\varphi(t,\omega)x\in A).$$

2.4 Invariant Measures for Random Dynamical Systems

An *invariant measure* for the random dynamical system φ over (Ω,ϑ) is a probability measure μ on $\Omega\times\mathbb{R}^d$ with marginal $\pi_\Omega\mu=\mathbb{P}$ on Ω, which is invariant under the skew product $\Theta(t)$.

If X is a standard measurable space (which is the case if X is a Borel subset of its completion), than μ has a disintegration $d\mu(\omega,x)=d\mu_\omega(x)d\mathbb{P}(\omega)$, where $(\mu_\omega)_{\omega\in\Omega}$ is a family of probability measures on X ($\mathbb{P}$–a.s. uniquely determined by μ). The invariance condition then means $\varphi(t,\omega)\mu_\omega=\mu_{\vartheta_t\omega}$ $\mathbb{P}$–a.s.

An ergodic invariant measure for a differentiable random dynamical system (i.e. X is a smooth finite dimensional manifold and $x\mapsto\varphi(t,\omega)x$ is differentiable) allows a "local" analysis of φ based on the multiplicative ergodic theorem of Oseledets ([33], see also Arnold [1, Theorem 4.2.6]), which provides a substitute of linear algebra. Exponential expansion rates for the linearized system (Lyapunov exponents, which generalize the real parts of eigenvalues; they are independent of ω and the initial value if $\mathbb{P}$ and μ are