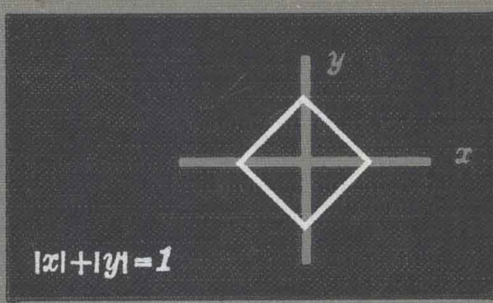
PRINCIPLES OF MATHEMATICS

Allendoerfer & Oakley



Principles of Mathematics

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PREFACE

This book has been written with the conviction that large parts of the standard undergraduate curriculum in mathematics are obsolete, and that it is high time that our courses take due advantage of the remarkable advances that have been made in mathematics during the past century. All other branches of science manage to incorporate modern knowledge into their elementary courses, but mathematicians hesitate to teach their elementary students anything more modern than the works of Descartes and Euler. It should be granted that mathematics is a cumulative subject and that one cannot run until he has learned to walk. Thus it is not realistic to start our students off with Functions of a Complex Variable, or other higher branches of our subject. The authors believe, however, that some of the content and much of the spirit of modern mathematics can be incorporated in courses given to our beginning students. This book is designed to do just that.

Since much of the material presented here has not commonly been taught without a substantial background on the part of the students, it has been necessary to experiment with methods of presentation. Our objective has been to tell as much of the subject as the students can comprehend, and to leave the higher development to later courses. We have repeatedly been surprised at the large amount of this material which can be taught just as easily as the standard material in the traditional courses. We have taught portions of this book in many preliminary forms to freshmen at Haverford College and to unselected undergraduates at the University of Washington. Our experience is that the material goes as well or better than that traditionally given.

The first four chapters are an introduction to abstract mathematical thinking, and we believe that their mastery greatly assists students in understanding the more usual topics which follow. Chapter 5 begins with an elementary treatment of sets, and then discusses Boolean Algebra. Applications of Boolean Algebra are then made to the theory of electric circuits in a form not readily available in elementary texts. Chapters 6 to 10 include what we believe useful and important in the usual courses in College Algebra, Trigonometry, and Analytic Geometry. Nevertheless, these chapters are not just a condensation of the usual material, but amount to a serious recasting of the basic ideas. After Chap. 10 a student is prepared to begin a standard course in Calculus.

Since most Calculus books are mainly concerned with the mechanics of the subject instead of the ideas involved, we have included Chaps. 11 and 12 as an introduction to the Calculus. The conceptual side is emphasized, and computations are restricted almost entirely to polynomials. It is our belief that students can progress much more rapidly in a standard Calculus course if they have mastered these two chapters first.

The final chapter, Chap. 13, concerns Statistics and Probability. In view of the importance of Statistics in recent years, it seems most unfortunate that our elementary students are not exposed to the elements of this subject. This is even more pitiful in view of the time spent on Classical Probability, for an introduction to Statistical Inference can be built upon the study of Elementary Probability without spending a large amount of additional time. This chapter has been placed last, but it can be studied at any time after Chap. 1.

The text was originally designed as a year course for freshmen at a liberal arts college. Students entering the course should have a working knowledge of Intermediate Algebra; and at the end of the course they should be prepared for a standard course in Calculus. There are, however, many other types of courses in which the book can be used. In particular, selected chapters can be used as a substitute for the usual College Algebra, and another selection of material is suitable for an introductory "general" course for nonspecialists. As an aid to teachers who may be considering the book as a text for any of these purposes we give the following suggested outlines of how the material can be used.

First, we give the number of class meetings in which each chapter can be covered. These figures do not include the appendixes which have been added at the ends of several chapters and which may be included or not according to the needs of the students. Further, they do not include time for review days and examinations. The lesson assignments are pitched for competent students, and less well-prepared individuals will undoubtedly have to proceed more slowly.

CHAPTER	NO. OF LESSONS
1. Logic	10
2. Number System	9
3. Groups	3
4. Fields	4
5. Sets and Boolean Algebra	6
6. Functions	8
7. Algebraic Functions	6
8. Trigonometric Functions	14
9. Exponential and Logarithmic Functions	4
10. Analytic Geometry	12
11. Limits	13
12. Calculus	8
13. Statistics and Probability	22

On this basis the following course outlines are among the many possibilities:

(1) Year course meeting five times a week (150 meetings)—entire book (including some appendixes).

(2) Year course meeting four times a week (120 meetings)—Chaps. 1 to 10 (inclusive) and either Chaps. 11 and 12 or Chap. 13.

(3) Year course meeting three times a week (90 meetings)—Chaps. 1 to 10 (inclusive).

(4) College Algebra Substitute (50 meetings)—Chaps. 1 to 7 (inclusive) with possible abbreviation of Chaps. 2 and 5. In Chap. 2, Secs. 6 to 16 can be omitted if desired, and in Chap. 5 everything following Sec. 5 can be omitted.

(5) "General" Liberal Arts Course. Selections from Chaps. 1 to 5 (inclusive), depending on the time available. It is possible to teach this material to students who have not studied Intermediate Algebra, but who have had the usual two years of Algebra and Plane Geometry in high school.

An unusual feature of the book is the set of references at the ends of the chapters. It is quite unusual for teachers of mathematics to assign outside reading to their students, and often the students get the impression that mathematics is all discovered and fully embalmed in their text. One reason for this, we are sure, is the scarcity of reference material which the students can read with profit. In order to correct this situation, we have listed not only other books but also carefully selected articles from the *American Mathematical Monthly* which can be read easily by the students. Further references to articles in this journal can be obtained by consulting the cumulative index to vols. 1 to 56 (inclusive), which was published in September, 1950, as part of vol. 57.

No standard tables have been included in the book. It is expected that students studying Chaps. 8, 9, and 13 will have suitable tables available to them in separate form. We have consistently referred to the Standard Mathematical Tables published by the Chemical Rubber Company, but other tables can be used equally well.

The more difficult exercises have been marked with asterisks (*) and can be omitted if desired. A few exercises are marked "BT," which means "Booby Trap," "Use your head," "Be careful," or "Don't make a fool of yourself."

The authors wish to acknowledge their debt to the writers of a number of pioneer textbooks which have treated similar material. We are also grateful to the many teachers who have sent us comments on the various preliminary editions. These helpful suggestions have enabled us to make many important improvements in our presentation. Further, we wish to thank the authorities at Haverford College and the University of Washington for their encouragement and help in preparing the preliminary editions. Finally, we wish to express our deep appreciation to Mrs. Jewel Kilgour whose accurate and rapid typing made the preparation of the manuscript possible.

It is hoped that the book is relatively free of errors, but each author blames the other for those which may be discovered.

CARL B. ALLENDOERFER
CLETUS O. OAKLEY

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LIST OF SYMBOLS

$\wedge$	and	16
$\vee$	or, inclusive	16
$\rightarrow$	implies	16
$\leftrightarrow$	equivalent to, for propositions	16
	for sets	105
$\sim$	not	16
$\exists_x$	for some x	16
$\forall_x$	for all x	16
$\underline{\vee}$	or, exclusive	20
$1.\overline{14}$	repeating decimal	53
$<$	less than	57, 96
$>$	greater than	57, 96
$\leq$	less than or equal to	57
$\geq$	greater than or equal to	57
$\|a\|$	absolute value	57, 131
(x,y)	coordinates of a point in the plane	58
$\equiv$	congruent to	67
$\circ$ in $a \circ b$	group operation	71
a'	group inverse of a	71
e	group identity	71
$\oplus$	sum of vectors	75
$\neq$	not equal to	80
$\subseteq$	inclusion	105, 111
$\subset$	proper inclusion	106
$\cap$	intersection	107
$\cup$	union	107
$\emptyset$	null set	107
$[a,b]$	the set $a \leq x \leq b$	107
$[a,b)$	the set $a \leq x < b$	109
$(a,b]$	the set $a < x \leq b$	109
(a,b)	the set $a < x < b$	109

A'	complement of set A	109
$f(x)$	value of the function f at x	126
$X \rightarrow Y$	notation for a function	127
$f:(x,y)$	notation for a function	127
$[\ ]$	greatest integer in	131
$\lim_{n \to \infty}$	limit of a sequence as n approaches infinity	292
$\lim_{x \to a} f(x)$	limit of the function $f(x)$ as x approaches a	303
Σ	summation	319
Δx	change in x	330
$\int_a^b f(x)\,dx$	integral of f	330
Δf	$f(x + \Delta x) - f(x)$	336
f'	derivative of f	336
$D_x f$	derivative of f	336
df/dx	derivative of f	336
$P_{n,n}$	number of permutations of n things, n at a time	384
$n!$	n factorial $= 1 \times 2 \times \cdots \times n$	384
$P_{n,r}$	number of permutations of n things, r at a time	385
$P_{n,n}^{r_1, r_2} \cdots$	number of permutations of n things, n at a time, of which r_1, r_2, . . . are alike	385
$C_{n,r}$	number of combinations of n things taken r at a time	388, 391
$P(E)$	probability of E	392
$P(E \wedge F)$	probability of both E and F	392
$P(E \mid F)$	probability of E, given F	392
$P(E \veebar F)$	probability of E or F, but not both	392
$P(E \vee F)$	probability of E or F or both	392

Chapter 1

LOGIC

1. Introduction

Many of the accomplishments of man are due to his ability to think logically in certain fields of activity, and many of his failures and disappointments stem from his lack of ability to do so in other fields. The physical sciences and particularly mathematics enjoy the reputation of being "logical," and successful applications of logic to other endeavors often earn the compliment of being conducted "with almost mathematical precision." Since the logical preeminence of mathematics is due to the peculiar nature of the subject rather than to the superior brilliance of mathematicians, it is necessary to begin the serious study of mathematics with an examination of the logical principles which underlie it. Once the application of logic to mathematics is clearly understood, the difficulty of its application to other fields will be diminished.

Our objective in this chapter is quite limited. We introduce only the most fundamental ideas of logic and make no pretense of building a complete logical system; the reader must turn elsewhere for such material. The argument is directed toward an understanding of the nature of mathematical proof, of the character of mathematical abstractions, and of the relationship of abstract mathematics to its concrete applications. The reader should not be misled into thinking that this chapter will give complete understanding of these matters. For this is only the beginning, and his understanding will grow as he meets the mathematical ideas to come in later chapters and as his mathematical and logical maturity grows.

2. Definitions

Long experience has shown that little progress in any argument can be attained unless all parties are agreed as to the meanings of the terms employed. Indeed, it is common practice for those who wish to prevent an agreement to twist the meanings of words purposely so that no logic can be applied to the situation. Thus it is necessary to have definitions, but how are they to be formulated?

Most people are used to looking to a dictionary for definitions of unfamiliar words. What they find, of course, is a statement of the meaning of the new word in terms of other words. But what do these words mean? If we look them up, they are defined in terms of still more words. If we carry this process on indefinitely, we shall eventually start going in circles such as:

dead: lifeless
lifeless: deprived of life
life: the quality distinguishing an animal or plant from inorganic or *dead* organic bodies

To take another example, we may define:

point: the common part of two intersecting *lines*
line: the figure traced by a *point* which moves along the shortest path between two *points*

The evils of this circular process can be seen by supposing that we try to learn a foreign language, say French, by using only an ordinary French dictionary—not a French-English dictionary. We look up a particular French word and find it described in more French words, and we find ourselves no farther ahead. Without a knowledge of a certain amount of French, a French dictionary is useless; and the same is true of our own English dictionary.

In this book we are starting to learn something about mathematics, and we shall soon find that mathematics has many similarities to a foreign language. We must learn what its words mean and must build up a basic mathematical vocabulary. The thing that will appear strange to the student is that in mathematics we will not allow our chain of definitions to become circular. We will take a small number of words which are *undefined.* Then other words are defined in terms of these undefined words, or in terms of words which themselves are defined in terms of the undefined words. This process is not in general use in most fields of knowledge, but it is essential for the construction of any logical system, and it is universally applied in mathematics.

It may come as a shock to some students to find that we do not define certain words, and indeed that we *cannot* define them; there-