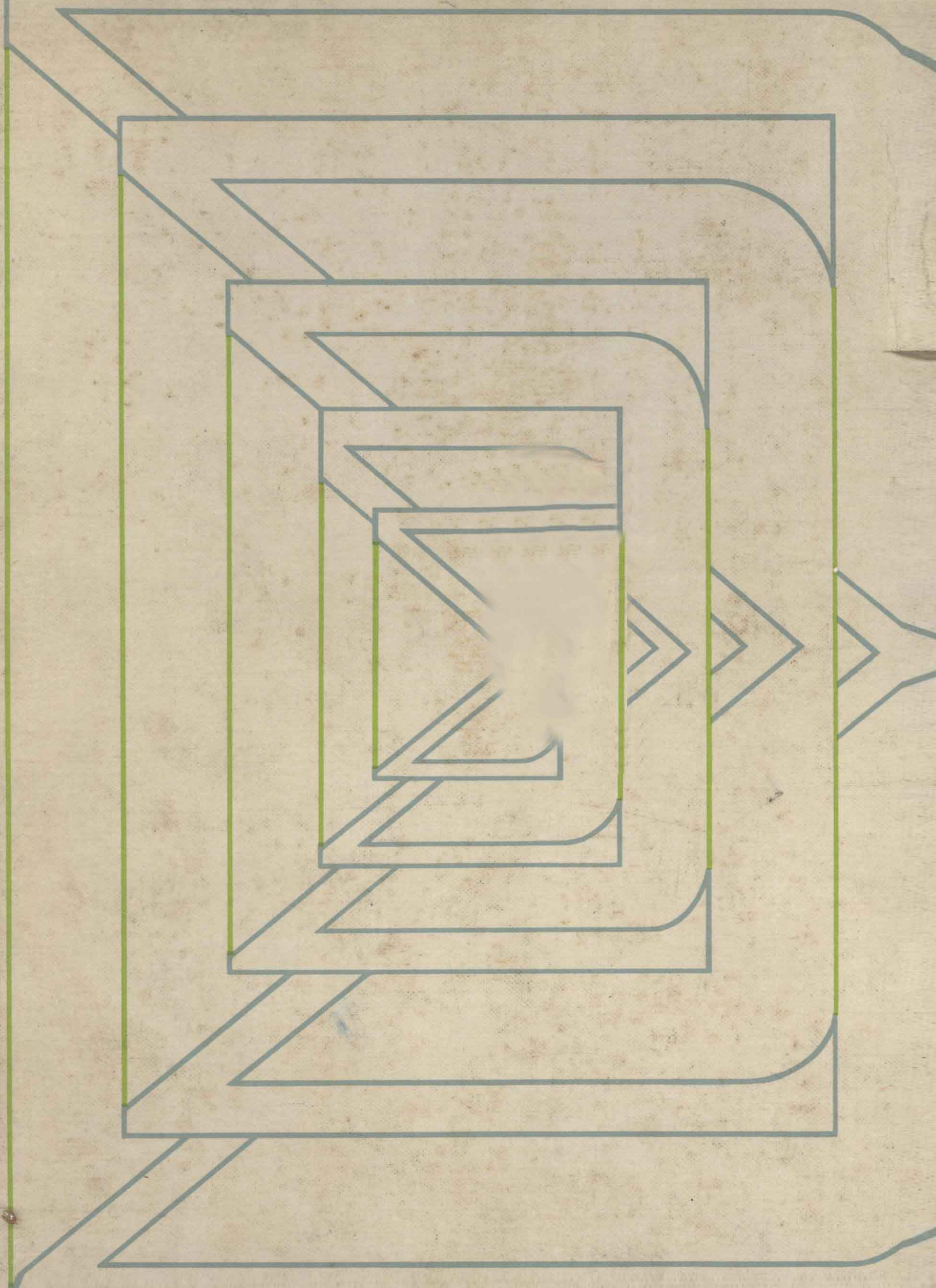

CALCULUS WITH ANALYTIC GEOMETRY

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PRENTICE-HALL, INC.
Englewood Cliffs, New Jersey

Current printing (last digit): *10 9 8 7 6 5 4 3 2*

13-112060-3

Library of Congress Catalog Card Number 71-101203

PRINTED IN THE UNITED STATES OF AMERICA

PRENTICE-HALL INTERNATIONAL, INC., *London*
PRENTICE-HALL OF AUSTRALIA, PTY. LTD., *Sydney*
PRENTICE-HALL OF CANADA, LTD., *Toronto*
PRENTICE-HALL OF INDIA PRIVATE LTD., *New Delhi*
PRENTICE-HALL OF JAPAN, INC., *Tokyo*

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Foreword

The debate over how to teach calculus is almost as old as the subject itself. Since World War II calculus enrollments have grown and diversified, reflecting university expansion and the increased use of mathematics in science, technology, and social science. The intensified dialogue about teaching can be seen in the accompanying plethora of calculus books. They give weighty testimony to the fact that there is no secret formula for teaching calculus.

There has been one very heartening development in recent years—the serious questioning of the nature and use of a mathematics textbook. We have discovered the student. We have come (rather late) to the realization that the basic format of a scholarly treatise may not mesh so well with the students' learning process. If a text is essentially a sequence of lectures in written form, then the plain and simple fact is that the average student doesn't read the text. He uses those parts of the book which provide something not found in his classroom lectures.

Burton Rodin has written a text based on a realistic picture of the day-to-day operations of many calculus instructors: they expound on the conceptual material and on some of the problems, and share with their students the constant frustration of insufficient time for the development of technical skills. Rodin's book contains a concise and rigorous development of calculus as an intellectual discipline. In larger type, it contains a parallel exposition which focuses on examples and technical skills in a style which the student can work his way through. The writing has the touch of a mathematician who is also a gifted and sensitive teacher. A considerable number of calculus teachers should see in this book the help which their students need.

Kenneth Hoffman
Cambridge, Massachusetts

Preface

ONE GOAL OF A CALCULUS COURSE is to teach routine problem solving and computational skills. There are other goals as well. A calculus course should impart an appreciation of the concepts and logical structure of the subject, and it should show how ingenuity and careful analysis are needed in handling nonroutine problems and applications.

The teaching of routine skills is the area in which a textbook can perform its greatest service. This frees the instructor to spend more time on the other goals mentioned above, where student-teacher interaction is so essential. Therefore I have included a large number of examples to illustrate routine problem solving, and I have taken great care to make them understandable to the student. In order to achieve this end about fifty per cent of these examples are written in a style approaching Socratic dialogue. This step-by-step, question-answer method is also intended to teach students how to read mathematics. Their progress in this area is tested by the final four chapters which are written in conventional style.

I have never found a formula for deciding, from a pedagogical point of view, which theorems of calculus should be proved rigorously and which ones should be presented only heuristically. The answer seems to depend on the week-to-week interests and attitudes of each individual class. For this reason I feel that the instructor alone can best decide when, and in what amounts, the logical structure of calculus should be explained to his students. I have presented a complete, rigorous development of these theoretical aspects from which the instructor can conveniently select what to explain in detail and what to describe only intuitively.

A large number of problems accompany each section. They range from routine drills to challenging problems involving the theory and applications of calculus. At the end of each chapter there is a collection of review problems. Problems which are especially difficult or which treat peripheral issues are designated with an asterisk ("starred problems"). Answers to all odd-numbered problems, often accompanied by the method of solution, are given following the problems.

This book begins immediately with integration rather than analytic geometry and differentiation (or set theory and the completeness of the real number system). The pedagogical advantages and disadvantages of this approach have been discussed by mathematicians for decades and no consensus has been reached. Personally, I finally came to favor this "integration first" approach because I know of no better way to arouse the student's enthusiasm for the subject than to show them, on the very first day of class, how to find the area under a parabola. In the succeeding weeks of the course further accomplishments, significant and new in the students' eyes, follow quickly. The students seem to develop a real feeling for the Riemann integral and its uses. They also develop a faith in the power of calculus which provides the motivation to carry them through subsequent topics which are more difficult and have less immediate uses. Another advantage of this approach is that it allows a gradual introduction to limits; thus the *limit of a sequence* is the only limit notion used in Chapter 1. Finally, it should be mentioned that this approach does no harm to students who require an early grasp of calculus techniques for use in their science courses. Since the user of this book gets into calculus immediately, he develops the techniques for doing physics problems involving work and acceleration faster than with most other texts.

This book covers all the material in a standard freshman-sophomore calculus course of three or four semesters. Linear algebra plays an essential role in the development, but always in a concrete setting; thus vectors are introduced in Chapter 6 and used consistently thereafter. Linear and affine transformations form the cornerstone of the treatment of multi-variable calculus in Chapter 12. A great deal of high school review material is included throughout the book. It occurs when needed, rather than in a preliminary chapter, so that the instructor can, if he wishes, conveniently omit it from class work yet assign it for homework. The chapters were arranged to treat the elementary aspects of multi-variable calculus (Chapters 7 and 8) before the single-variable topics of Differential Equations and Infinite Series (Chapters 10 and 11). If the instructor wishes to cover all of the single-variable material before introducing functions of several variables, this can be done by skipping Chapters 7 and 8, returning to them immediately before beginning Chapter 12.

The final form of the example problems evolved from two years of class testing. I wish to thank the following instructors who read or tested various drafts of the manuscript and offered significant improvements: Professors Terence Butler (Rutgers University), Larry Goldstein (Yale University), Jerry Kazdan (University of Pennsylvania), Ed Landesman (University of California), Alfred Manaster (University of California), George Pedrick (Purdue University), Dave Ragozin (University of Washington), Nick Rose (North Carolina State University), Rick Travis (Palm Beach Jr. College), Jim Wahab (University of South Carolina), Paul Weichsel (University of Illinois), Gill Williamson (University of California).

I was very fortunate during the writing of this book to have the help of a distinguished teacher, Professor Herbert Gindler of San Diego State College. All the good ideas in this book are an outgrowth of stimulating discussions we had together. The wisdom and wit of Professor Kenneth Hoffman of M.I.T. improved the manuscript immeasurably—it was a great pleasure to work with him. James Walsh and Robert Martin of Prentice-Hall contributed many excellent suggestions; I am grateful to them for their inspired handling of this project. To Diane Dickey and Edward Hendricks go my thanks for working the problems. I am indebted to Juanita Rossé who typed a perfect manuscript.

Burton Rodin
La Jolla, California

Contents

1

THE RIEMANN INTEGRAL

Summary: The first two sections show how to calculate, using a limit process, areas bounded by the graphs of linear and quadratic polynomials. This leads to the precise notions of ***limit of a sequence, Riemann sum,*** *and* ***Riemann integral*** *given in Section 1-3. In Section 1-4 we derive the formula for integrating x^n and formulas concerning the linearity of integrals. These formulas are applied to the calculation of areas in Section 1-5, and of volumes in Section 1-6. The final Section 1-7 treats some properties and uses of continuous functions.*

SECTION 1-1
APPROXIMATING AREAS

Summary: ***In this section we introduce the problem of finding the area between the graph of a nonnegative function $y = f(x)$ and the x-axis, for x in an interval $[a, b]$. Although the problem is not solved completely, an approximate solution is obtained which the student learns to write compactly in the form***

$$\sum_{i=1}^{n} f(\bar{x}_i)(x_i - x_{i-1}).$$

Special Formulas (1-2), (1-3) are given to make this expression easy to evaluate if f is a linear or quadratic polynomial. The concepts of function, graph, and $\sum$-notation are reviewed.

1A *Real numbers*

DISCUSSION Nearly everything in calculus involves numbers. Let us briefly discuss the properties you should know. You are expected, of course, to be familiar with the arithmetic of the *integers*

$$\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots.$$

The *rational numbers* (or *fractions*) consist of all numbers of the form p/q where p and q are integers ($q \neq 0$). You are also expected to be familiar with the arithmetic of rational numbers, for example,

$$\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd}, \qquad \left(\frac{a}{b}\right)\left(\frac{c}{d}\right) = \frac{ac}{bd}, \qquad \frac{\frac{a}{b}}{\frac{c}{d}} = \frac{ad}{bc}.$$

There are some numbers such as $\sqrt{2}$, $\sqrt[3]{5}$, π, etc., which are not rational numbers (see Problem 19 on page 90); they are called *irrational numbers.*

Observe that the rational numbers include the integers; for example $5 = 5/1$, $-2 = -2/1$, etc. The totality of all rational and irrational numbers form the system of *real numbers.* Informally, the real number system is often described as the set of all decimal numbers, including those with infinitely long decimal expansions; for example,

$$\tfrac{4}{3} = 1.33333\ldots, \qquad \pi = 3.14159\ldots, \qquad 2 = 2.00000\ldots.$$

What do we expect you to know about the real number system? First, we assume you are familiar with the device of representing real numbers by a number axis (see FIGURE 1-1). Note that we follow the convention of placing positive numbers to the right of 0 and negative numbers to the left. We assume that you are familiar with the *order relation* $x < y$ (read "x is less than y"); thus $0 < 1$, $2 < 3$, $-2 < 1$, etc. Finally, we assume that

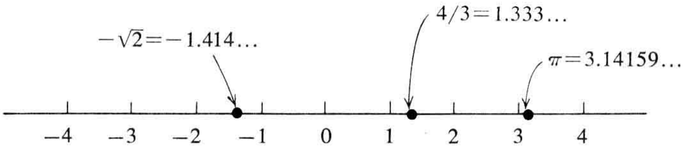

Figure 1-1

you are familiar with the basic rules for manipulating algebraic expressions and inequalities — we shall review these rules in detail as we need them (Sections 1-3, 2-4).

In order to develop calculus rigorously it is necessary to state at the outset precisely what properties of the real number system are assumed to be true. First of all, we assume the simple algebraic properties listed on page 710 (the Field Axioms). Secondly, we assume certain properties of the order relation $x < y$. The precise properties we assume are:

(i) If $x < y$ and $y < z$ then $x < z$.
(ii) If $x < y$ then $x + z < y + z$.
(iii) If $x < y$ and $0 < z$ then $xz < yz$.
(iv) Given x and y then exactly one of the following relations holds: $x < y$, $y < x$, or $x = y$.

We make use of the notations

$x \leqq y$	means	$x < y$ or $x = y$,
$x > y$	means	$y < x$,
$x \geqq y$	means	$x > y$ or $x = y$,
$x \leqq y \leqq z$	means	$x \leqq y$ and $y \leqq z$,
$x < y < z$	means	$x < y$ and $y < z$.

If $x > 0$ we say x is *positive;* if $x \geqq 0$ we say x is *nonnegative.* In Section 1-3 we give a more detailed review of this subject of inequalities.

Thirdly, we shall make use of the so-called *Completeness Property* of the real number system. You are not expected to be familiar with this property now. The full force of it will not be needed until Section 1-7, and so we shall postpone the statement of it until then. Meanwhile, we must assume the following special case of the Completeness Property:

If n is a positive integer and if r is a positive number, then there is a unique positive number x such that $x^n = r$; we denote this number by $\sqrt[n]{r}$.

It should be pointed out that on this theoretical level we need not assume any properties of the geometric representation of numbers by a number axis. The reason is that calculus can be *developed* rigorously without using any notions from geometry. From the viewpoint of pure logic such a development is to be preferred, since the number of assumptions made is thereby kept to a minimum. Nevertheless, geometric interpretations and applications are extremely important for *understanding* calculus, and so we shall use them quite often.

1 B *Functions and graphs*

Let us briefly discuss the concept of function. From now on the word "number" will be used to mean "real number."

The abstract definition of function is this. Consider a collection of ordered pairs of numbers; if (x, y) is such a pair, then we call x the first element of the pair and y the second element. A collection of such ordered pairs is called a *function* if it does not contain two pairs which have identical first elements and distinct second elements; that is, it does not contain two pairs (x, y) and (x, y') where $y \neq y'$. Given such a collection, we define its *domain* to be the set of all numbers x which appear as the first element of some ordered pair in that collection.

If we are given a function and if x is a number in its domain, then there is a definite number y such that (x, y) belongs to the function. We may express this by saying that the function provides a rule for assigning to x the number y. This point of view justifies the following common description of function: *Let D be a set of numbers. A rule which assigns to each number in D some definite number is called a function, and D is called the domain of that function.*

Let the letter f denote a function with domain D. If x is a number in D, then f assigns a definite number to x. That number which f assigns to x is denoted by $f(x)$ [the symbol $f(x)$ is read as "f of x"].

EXAMPLE *1-1* Let f be the function which assigns to each number its square. Then f assigns to -3 the value 9. In symbols we write $f(-3) = 9$. In general, to the number x the function f assigns the value x^2; in symbols, $f(x) = x^2$. The domain of f is the set of all numbers because when we defined f we said that it assigns to *each* number its square. Let us discuss the reason f is really a function. To qualify as a function this f must assign to the number x one and only one value. Since x^2 is a number and its value is definitely determined by x, this rule f is actually a function. (If you want to see a similar rule which does not define a function, look at Example 1-2.) Answer the following questions concerning this function f. See page 728 for the correct answers. (A) $f(1) =$ ____, $f(-1) =$ ____, $f(0) =$ ____, $f(\sqrt{2}) =$ ____, $f(t) =$ ____. (B) What is $f(x^3)$? (C) $f(a + 3) =$ ________. (D) Is $f(3) + f(5) = f(3 + 5)$? (E) $f(a + b) - [f(a) + f(b)] =$ ____. (F) For what values of x do we have $f(x) = 7$? $f(x) = -7$? (G) $f(f(4)) =$ ____, $f(f(x)) =$ ____. (H) $f(0) + f(1) + \cdots + f(6) =$ ________.
(I) $f(\frac{1}{4}) \cdot \frac{1}{4} + f(\frac{1}{2}) \cdot \frac{1}{4} + f(\frac{3}{4}) \cdot \frac{1}{4} + f(1) \cdot \frac{1}{4} =$ ________.
(J) $\dfrac{f(x + h) - f(x)}{h} =$ ____, $(h \neq 0)$.

EXAMPLE *1-2* We now illustrate a rule which does not define a function. Let g assign to each nonnegative number x any number whose square is x. What value does g assign to the number 4? If your answer is 2 you are correct. If your answer is -2 you are correct. Thus g does not assign a *definite* value to each number in its domain. Hence g is not a function. Note that the equation $g(x) = \sqrt{x}$ does not describe the rule g correctly, since $\sqrt{x}$ always means the nonnegative square root of x. It would, however, be correct to write $g(x) = \pm\sqrt{x}$.

EXAMPLE *1-3* Often a rule is defined by simply writing down an algebraic expression involving one unknown quantity. This unknown is called a variable. When we use this method to define a function, let us always agree that the domain of the function shall consist of all numbers for which the rule makes sense.

For example, let G be the rule $G(x) = \sqrt{x}$. This means that the rule G assigns to x the nonnegative square root of x. This rule makes sense only if x is a nonnegative number, since $\sqrt{x}$ is not a number if x is negative. Thus by our agreement above, the domain of G consists of all nonnegative numbers. (A) Is G a function? (B) $G(0) =$ ____, $G(4) =$ ____, $G(t) =$ ____. (C) What is $G(a^2)$ (careful!)? (D) Is $G(3^2 + 4^2) = 3 + 4$? (E) $G(G(G(16))) =$ ____. (F) Let $f(x) = x^2$. What is $f(G(5))$? What is $G(f(-2))$? (G) By rationalizing the numerator, simplify $\dfrac{G(x+h) - G(x)}{h}$, $(h \neq 0)$ ["rationalizing the numerator" means to multiply the numerator and denominator by $G(x+h) + G(x)$].

EXAMPLE *1-4* If x is any number, let $[x]$ denote the largest integer which is less than or equal to x. Thus $[\pi] = 3$, $[5.6] = 5$, $[5] = 5$, $[-1.5] = -2$. If we define a rule f by $f(x) = [x]$, then f is a function. By our agreement, its domain consists of all numbers. This function f cannot be expressed by means of a simple algebraic expression involving x. During this course we shall come across many other functions which cannot be defined by algebraic expressions.

EXAMPLE *1-5* (In this example we present a function which is expressed by two separate algebraic expressions.) A swimming pool is shaped so that it is 30 feet wide, 45 feet long, 2 feet deep at the shallow end, and 7 feet deep at the deep end, the bottom being an inclined plane. (A) Sketch the shape of the pool and draw in the proper dimensions.

The volume V (in cubic feet) of water in the pool is a function of the height h (in feet) of the water level above the deepest point of the pool. Thus $V(h)$ is the number of cubic feet of water in the pool when the surface of the water is h feet above the bottom. Or to put it another way, V is the rule which assigns to a number h the volume of water in this pool when the depth of water at the deepest part of the pool is h. In this example the domain of V is the set of numbers h with $0 \leqq h \leqq 7$. (B) Find the numerical value of $V(5)$ by discovering on your own whatever formulas you may need for the volumes of geometric solids. (C) If $5 < h \leqq 7$, find a formula for $V(h)$.

Using similar triangles, show that if $0 \leqq h \leqq 5$ then

$$V(h) = (\tfrac{1}{2})(30)\left(45 \cdot \frac{h}{5}\right)(h) \qquad \text{(in cubic feet).}$$

The function V can be described in the form

$$V(h) = \begin{cases} 135h^2 & \text{if } 0 \leqq h \leqq 5 \\ 1350h - 3375 & \text{if } 5 \leqq h \leqq 7. \end{cases}$$