Mathematics and Its Applications

Lev V. Sabinin

Mirror Geometry of Lie Algebras, Lie Groups and Homogeneous Spaces

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Когда раздвоился Хаос
До корня вздыбленных волос,
Взорвался Сингулярий–Конус
И начал бег весёлый Хронос,
Ехидна злая, Энтропия,
Сплела покровы роковые.
Родились Смерть, Распад и Тлен.
Застой и Мрак без перемен.

Но День Брамана уж на склоне,
Грядёт, бредёт Брамана Ночь,
Корова Дхармы ноги клонит,
Не в силах дрёму превозмочь.
Опустошённый, мир унылый
Устало завершает круг,
И Хронос замирает вдруг,
Хаос играет новой силой.
И Жизнь и Смерть,
И Мрак и Тлен,
В круговороте перемен.

Квантасмагор

L. Tobiriz - 20 January 1996. Morelia. Mexico.

On the artistic and poetic fragments of the book

It is evident to us that the way of writing a mathematical treatise in a monotonous logically-didactic manner subsumed in the contemporary world is very harmful and belittles the greatness of Mathematics, which is authentic basis of the Transcendental Being of Universe.

Everyone touched by Mathematical Creativity knows that Images, Words, Sounds, and Colours fly above the ocean of logic in the process of exploration, constituting the real body of concepts, theorems, and proofs. But all this abundance disappears and, alas, does so tracelessly for the reader of a modern mathematical treatise.

Therefore the attempts at attracting a reader to this majestic Irrationality appear to be natural and justified.

Thus the inclusion of poetic inscriptions into mathematical works has already (and long ago) been used by different authors. Attempts to draw (not to illustrate only) Mathematics is already habitual amongst intellectuals. In this connection let us note the remarkable pathological-topological-anatomical graphics of the Moscow topologist A.T. Fomenko.

In our treatise we also make use of graphics and drawings (mental images-faces of super-mathematical reality, created by sweet dreams of mirages of pure logic) and words (poetic inscriptions) which Eternity whispered during our aspirations to learn the beauty of Irrationality.

The reader should not search for any direct relations between our artistic poetic substance and certain parts and sections of the treatise. It is to be considered as some general super-mathematical-philosophical body of the treatise as a whole.

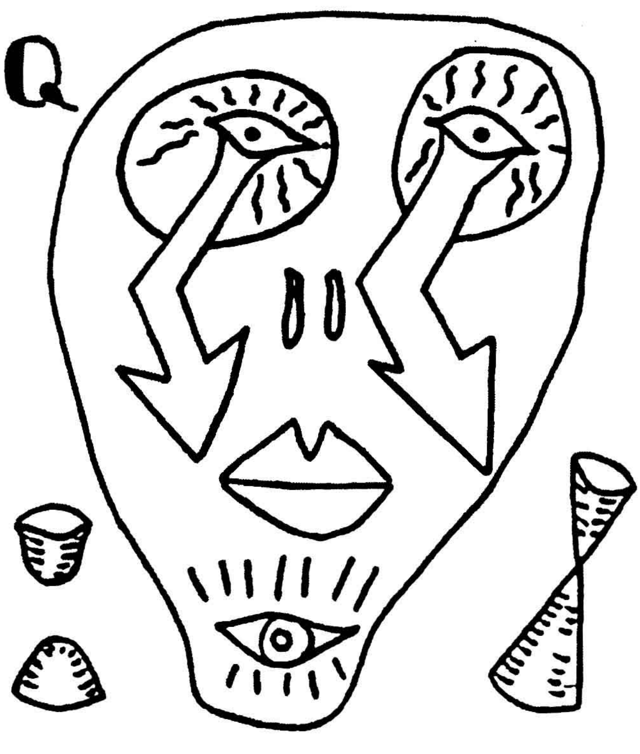
Lev Sabinin

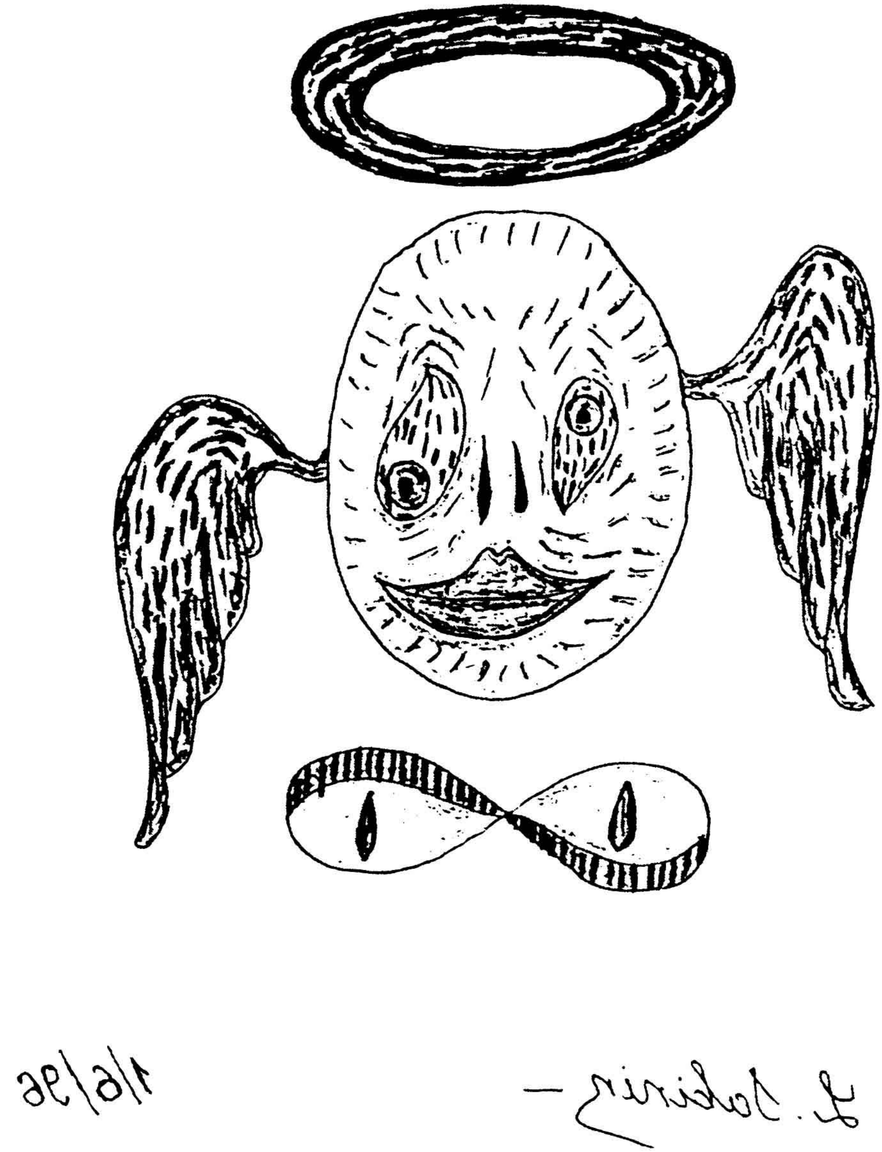
INTRODUCTION

Сфероид медленно вращался
Сметая сонмы странных Числ...
Миры, Века, катились вниз
И лик Творца обозначался.

Стремясь постигнуть тайный смысл
Сметенья стонов странных Числ,
Значков причудливый альянс,
Я погружался в сонный Транс.
Гримассы Жуть и Боли Смысл
На лицах прочитал я Числ.

Квантасмагор





INTRODUCTION

As K. Nomizu has justly noted [K. Nomizu, 56], Differential Geometry ever will be initiating newer and newer aspects of the theory of Lie groups. This monograph is devoted to just some such aspects of Lie groups and Lie algebras.

New differential geometric problems came into being in connection with so called subsymmetric spaces, subsymmetries, and mirrors introduced in our works dating back to 1957 [L.V. Sabinin, 58a,59a,59b].

In addition, the exploration of mirrors and systems of mirrors is of interest in the case of symmetric spaces. Geometrically, the most rich in content there appeared to be the homogeneous Riemannian spaces with systems of mirrors generated by commuting subsymmetries, in particular, so called tri-symmetric spaces introduced in [L.V. Sabinin, 61b].

As to the concrete geometric problem which needs be solved and which is solved in this monograph, we indicate, for example, the problem of the classification of all tri-symmetric spaces with simple compact groups of motions.

Passing from groups and subgroups connected with mirrors and subsymmetries to the corresponding Lie algebras and subalgebras leads to an important new concept of the involutive sum of Lie algebras [L.V. Sabinin, 65].

This concept is directly concerned with unitary symmetry of elementary particles (see [L.V. Sabinin, 95,85] and Appendix 1).

The first examples of involutive (even iso-involutive) sums appeared in the exploration of homogeneous Riemannian spaces with $ds^2 > 0$ and axial symmetry. The consideration of spaces with $(n-1)$-dimensional mirrors [L.V. Sabinin, 59b] again led to iso-involutive sums.

The construction of the so called hyper-involutive decomposition (sum) can be dated back to 1960–62, see, for example, the short presentation of our report at the International Congress of Mathematicians (1962, Stockholm) in volume 13 of Transactions of the Seminar on Vector and Tensor Analysis (1966, Moscow University) and [L.V. Sabinin, 67].

Furthermore, a very important heuristic role was played by the work of Shirokov [P.A. Shirokov, 57], in which the algebraic structure of the curvature tensor of the symmetric space $SU(n+1)/U(n)$ was given, and by the work of Rosenfeld [B.A. Rosenfeld 57]. That allowed us to construct characteristic iso-involutive decompositions for all classical Lie algebras [L.V. Sabinin, 65, 68].

In this way the apparatus for direct exploration of symmetric spaces of rank 1 with compact Lie groups of motions was introduced (avoiding the well known indirect approach connected with the Root Method and the examination of E. Cartan's list of all symmetric spaces with compact simple Lie groups of motions).

The most difficult, certainly, and fundamental element of the suggested theory was the understanding (1966) of the role of principal unitary and special unitary automorphisms of Lie groups and Lie algebras [L.V. Sabinin, 67,69,70]. The above work solved the problem of introducing the 'standard' mirrors into a homogeneous Riemannian space with $ds^2 > 0$.

Indeed, in this case the stationary subgroup is compact and, taking its principal unitary involutive automorphism (which is possible, except for trivial subcases), we can generate a 'standard' subsymmetry and a 'standard' mirror in a Riemannian space. Analogously, one can introduce systems of 'standard' mirrors in a Riemannian space with $ds^2 > 0$ and, furthermore, with their help, explore geometric properties of homogeneous Riemannian spaces.

The detailed consideration of involutive sums of Lie algebras has shown, however, that their role is more significant than the role of the only convenient auxiliary apparatus for solving some differential-geometric problems. We may talk about the theory of independent interest and it is natural to call it 'Mirror geometry of Lie algebras', or 'Mirror calculus'; the role and significance of which is comparable with the role and significance of the well known 'Root Method' in the theory of Lie algebras.

Part I and II of this treatise are devoted to the presentation of Mirror Geometry over the reals.

A Lie algebra $\mathfrak{g}$ has the group of automorphisms $\mathrm{Aut}(\mathfrak{g})$ and consequently generates the geometry in the sense of F. Klein. In the case of a semi-simple compact Lie algebra $\mathfrak{g}$ the group $\mathrm{Int}(\mathfrak{g}) \subset \mathrm{Aut}(\mathfrak{g})$ is a compact linear Lie group, which allows us to use some knowledge from the theory of compact Lie groups (however, we need not too much from that theory, and necessary results can be proved without the above theory). We may regard Cartan's theorem on the existence of non-trivial inner involutive automorphism of a simple compact non-one-dimensional Lie algebra as the typical theorem of the Lie algebras geometry (the proof follows immediately from the existence of non-trivial involutive elements in $\mathrm{Int}(\mathfrak{g})$).

In an ordinary Euclidean space a plane can be defined as a set of all points immobile under the action of some involutive automorphism. Thus the maximal subset of elements immobile under the action of an involutive automorphism, that is, some involutive subalgebra $\mathfrak{l} \subset \mathfrak{g}$ in a compact Lie algebra, may be regarded as an analogue of a plane in an Euclidean space.

Let us now consider the problem of a canonical base of a compact Lie algebra. From the point of view of Classical Invariants' theory the problem of the classification of Lie algebras is connected with the finding of a base in which the structure tensor has a sufficiently simple form (canonical form).

In order to clarify what has just been said, let us consider a simple problem of that kind, namely, the problem of a canonical form for a bilinear form in a centered-Euclidean space. As is easily seen, here the determination of a canonical base is reduced to the finding of commuting isometric involutive automorphisms and the subsequent choice of a base in such a way that the above involutive automorphisms have the basis vectors as proper vectors. For this it is enough to find the 'standard' involutive automorphisms (for example, connected with reflections with respect to hyperplanes) of that type; other involutive automorphisms can be obtained as products of the 'standard' involutive automorphisms.

Any two commuting 'standard' involutive automorphisms S_1, S_2 generate the third automorphism $S_3 = S_1 S_2 = S_2 S_1$ (non-standard, in general) and consequently a discrete commutative group $\{\mathrm{Id}, S_1, S_2, S_3\}$, the so called involutive group $\chi(S_1, S_2, S_3)$.

Returning to a compact Lie algebra $\mathfrak{g}$ we see that the construction described above is valid here, and in a natural way we have the notion of involutive group $\chi(S_1, S_2, S_3) \subset \mathrm{Aut}(\mathfrak{g})$.

The only problem in the consideration presented above is to introduce, reasonably, the 'standard' involutive automorphisms for any compact semi-simple Lie algebra.

With any involutive group $\chi(S_1, S_2, S_3)$ of a Lie algebra $\mathfrak{g}$ one may associate in a natural way the decomposition

$$\mathfrak{g} = \mathfrak{l}_1 + \mathfrak{l}_2 + \mathfrak{l}_3 \,, \qquad \mathfrak{l}_1 \cap \mathfrak{l}_2 = \mathfrak{l}_2 \cap \mathfrak{l}_3 = \mathfrak{l}_3 \cap \mathfrak{l}_1 = \mathfrak{l}_0 \,,$$

where $\mathfrak{l}_1, \mathfrak{l}_2, \mathfrak{l}_3$ are involutive algebras of the involutive automorphisms S_1, S_2, S_3, respectively, ($\mathfrak{l}_\alpha = \{\zeta \in \mathfrak{g} \mid S_\alpha \zeta = \zeta\}$). This is a so called involutive decomposition (involutive sum). As well, one can introduce the corresponding involutive base (in fact, a set of involutive bases) whose vectors are proper vectors for S_1, S_2, S_3. Thus if we are interested in a canonical base of the Lie algebra $\mathfrak{g}$ then it is an involutive base of some involutive group.

Among involutive groups $\chi(S_1, S_2, S_3)$ one may select two special classes, namely: iso-involutive groups, $\chi(S_1, S_2, S_3; \varphi)$, where S_1 and S_2 are conjugated by $\varphi \in \mathrm{Aut}\,\mathfrak{g}$, $\varphi^2 = S_3$, and hyper-involutive groups, $\chi(S_1, S_2, S_3; p)$, where S_1 and S_2, S_2 and S_3, S_3 and S_1 are conjugated by $p \in \mathrm{Aut}\,\mathfrak{g}$. They generate, respectively, iso-involutive sums, iso-involutive bases and hyper-involutive sums, hyper-involutive bases.

We show that any arbitrarily taken non-trivial simple compact Lie algebra $\mathfrak{g}$ ($\dim \mathfrak{g} \neq 1$) with an involutive automorphism S_1 has iso-involutive groups $\chi(S_1, S_2, S_3; \varphi)$.

This result turns iso-involutive sums into an instrument of exploration of Lie algebras.

Hyper-involutive sums are not universal to the same extent, but in appropriate cases serve also as an effective apparatus of investigation. One sufficient condition for the existence of hyper-involutive sums for a simple compact Lie algebra $\mathfrak{g}$ is: there exists a three-dimensional simple subalgebra $\mathfrak{b} \subset \mathfrak{g}$ such that the restriction $\mathrm{Int}_{\mathfrak{g}}\mathfrak{b}$ of $\mathrm{Int}\mathfrak{g}$ to $\mathfrak{b}$ is isomorphic to $SO(3)$.

Now we pass to the problem of determination of 'standard' involutive automorphisms of a simple compact Lie algebra $\mathfrak{g}$ ($\dim \mathfrak{g} \neq 1$).

An involutive algebra $\mathfrak{l} \subset \mathfrak{g}$ (and the corresponding involutive automorphism S) is called principal if it contains a simple three-dimensional ideal $\mathfrak{b}$, that is, $\mathfrak{l} = \mathfrak{b} \oplus \tilde{\mathfrak{l}}$. In this case, if $\mathrm{Int}_{\mathfrak{g}}(\mathfrak{b}) \cong SO(3)$ then we say that $\mathfrak{l}$ (and S) is principal orthogonal and if $\mathrm{Int}_{\mathfrak{g}}(\mathfrak{b}) \cong SU(2)$ then we say that $\mathfrak{l}$ (and S) is principal unitary.

By means of involutive decompositions we prove the main theorem: any simple compact Lie algebra $\mathfrak{g}$ ($\dim \mathfrak{g} \neq 1$) has a non-trivial principal involutive automorphism. If $\dim \mathfrak{g} \neq 3$ then $\mathfrak{g}$ has a non-trivial principal unitary involutive automorphism.

This solves the problem of introducing 'standard' involutive automorphisms which may be regarded as principal.

One may introduce also the broader class of special involutive algebras and involutive automorphisms, in particular, the unitary special involutive algebras and involutive automorphisms.

We give, furthermore, the simple classification of principal unitary involutive automorphisms: principal di-unitary, principal unitary central, principal unitary of index 1, exceptional principal unitary. Using the apparatus of involutive sums and involutive bases we explore all these types. As a result the type of principal unitary involutive automorphism defines, in general, the type of simple compact Lie algebra. For example, if a simple compact Lie algebra has a principal unitary non-central involutive automorphism of index 1 then $\mathfrak{g}$ is isomorphic to $sp(n)$, $n > 1$. For the Lie algebra $\mathfrak{g}_2$ the construction is presented in a hyper-involutive base up to the numerical values of structural constants, that is, the problem is completely solved in the sense of the classical theory of invariants. For other types of exceptional Lie algebras we determine the basis involutive sums and the structure of their involutive algebras.

Furthermore, we consider the problem of the classification of special unitary non-principal involutive authomorphisms. The principle of the involutive duality of principal unitary and special unitary non-principal involutive automorphisms is established. This principle allows us to define all special simple unitary subalgebras and all special unitary involutive automorphisms for simple compact Lie algebras.

For all simple compact Lie algebras $\mathfrak{g}$ ($\dim \mathfrak{g} \neq 1$), except

$$so(3) \cong su(2) \cong sp(1)\,, \qquad so(5) \cong sp(2)\,, \qquad su(3)\,,$$
$$so(6) \cong su(4)\,, \qquad so(7)\,, \qquad so(8)\,, \qquad \mathfrak{g}_2\,,$$

we construct the basis iso-involutive decomposition $\mathfrak{g} = \mathfrak{l}_1 + \mathfrak{l}_2 + \mathfrak{l}_3$, where $\mathfrak{l}_1$ and $\mathfrak{l}_2$ are principal unitary involutive Lie algebras and $\mathfrak{l}_3$ is a special unitary involutive Lie algebra. By the type of such involutive sum the type of $\mathfrak{g}$ is uniquely defined. For each of the Lie algebras which have been excluded above we construct the basis hyper-involutive decomposition which uniquely characterizes any of them.

Furthermore, for simple compact Lie algebras we consider the possibility of constructing hyper-involutive sums with principal unitary involutive automorphisms.

Using the procedure of involutive reconstruction of basis involutive sums we prove that principal unitary hyper-involutive sums exist and are unique for Lie algebras $so(n)$ $(n > 5)$, $su(n)$ $(n > 2)$, $\mathfrak{f}_4$, $\mathfrak{e}_6$, $\mathfrak{e}_7$, $\mathfrak{e}_8$ (all these involutive sums are found) and do not exist for $sp(n)$.

It is shown that for $sp(n)$ $(n > 2)$ one can construct hyper-involutive sums with special unitary involutive algebras. An analogous construction is valid for $so(n)\,(n > 8)$, $su(n)$ $(n > 4)$, $\mathfrak{f}_4$, $\mathfrak{e}_6$, $\mathfrak{e}_7$, $\mathfrak{e}_8$. All such involutive sums are found as well.

Thus the suggested theory is a theory of structures of a new type for compact real Lie algebras and is related to discrete involutive groups of automorphisms and the corresponding involutive decompositions.

Let us now turn to possible geometric applications, which, in particular, may be found in Part III and IV.

First of all we note that since we deal with involutive automorphisms, all, or almost all, proved results may be reformulated in terms of symmetric spaces [E. Cartan, 49,52], [B.A. Rosenfeld, 57], [L.V. Sabinin, 59c], [S. Helgason, 62,78], [A.P. Shirokov, 57] and in terms of mirrors in homogeneous spaces.

Such applications are concentrated at the beginning of Part III after some necessary definitions. Despite that here many results have been obtained simply as a reformulation of theorems of Part I and II from the language of Lie algebras into the language of Lie groups, homogeneous spaces, and mirrors, those are very interesting. (For example, the characterization of symmetric spaces of rank 1 by the properties of geodesic mirrors.)

In addition, the theory of Part I and II implies two new interesting types of symmetric spaces—principal and special—and allows us to explore geometric properties of their mirrors.

Furthermore, in Section III.5 we consider applications of Mirror Geometry to some problems of simple compact Lie groups. Thus it is shown that $\mathrm{Int}(\mathfrak{g}_2)$ and $\mathrm{Int}(\mathfrak{f}_4)$ are the only simple compact connected Lie groups of types G_2 and F_4, respectively.

Moreover, it is shown how, knowing involutive decompositions for simple compact Lie algebras, one may find out their inner involutive automorphisms (in the cases of $\mathfrak{g}_2$, $\mathfrak{f}_4$, $\mathfrak{e}_6$, $\mathfrak{e}_8$).

Lastly, the final sections of Part III (III.6–III.8) are devoted to the complete classification of tri-symmetric spaces with a simple compact Lie group of motions. The solution of this problem, when treated by conventional methods, had serious difficulties. Indeed, the first part of this problem, the definition of involutive groups $\{\mathrm{Id}, S_1, S_2, S_3\}$ of automorphisms, is already not trivial. Since, even if all S_α are inner automorphisms, they can not be generated by a one Cartan subgroup, it is necessary to bring into consideration the normalizers of maximal tori ([Seminar Sophus Lie, 62] Ch. 20). But the determination of normalizers of maximal tori in exceptional simple compact Lie groups is a complicated problem owing to the absence of good matrix models. However, the theory developed in Part I and II gives a natural apparatus for solving the above problem.

The classification shows, in particular, that all non-trivial non-symmetric tri-symmetric spaces have isomorphic basis mirrors (in hyper-symmetric decomposition) and have irreducible Lie groups of motions if they are maximal. Their mirrors possess remarkable geometric properties being either principal or central. In this relation we note that in [O.V. Manturov, 66] two spaces, $G_2/SU(3)$ and $E_7/F_4 \times SO(3)$, with irreducible groups of motions have not been found.

The results of Part II belong to the area in which strong methods and detailed theories existed earlier. Therefore we naturally need some comparisons.

The theory of compact Lie algebras has been established mainly by the work of Lie [S. Lie, 1888,1890,1893], Killing [W. Killing, 1888,1889a,1889b,1890], E. Cartan [E. Cartan, 49,52], H. Weyl [H. Weyl, 25, 26a,b,c, 47], Van der Warden [B.L. Van der Warden 33], Dynkin [E.B. Dynkin, 47], Gantmacher [Gantmacher 39a,b] *etc.*, and is well known.

We now intend to compare the well known 'Root Method' with our new theory, which we briefly call 'Mirror Geometry'.

First of all, Mirror Geometry deals with new types of structures (involutive sums) and is introduced independently of the Root Method. Thus these two theories seem to be different. But since Mirror Geometry leads us to the classification of simple compact Lie algebras (through the classification of principal unitary involutive automorphisms) we need some comparisons.

The Root Method has a complex nature and the classification of real simple compact Lie algebras require the supplementary theory (the theory of real forms). Mirror Geometry has a real nature.

The determination of involutive automorphisms of Lie algebras in the Root Method requires a supplementary theory. In Mirror Geometry, owing to the construction, any type of simple compact Lie algebra appears together with two (generally speaking) involutive automorphisms, principal unitary and its dual, special unitary. This is of importance for exceptional Lie algebras (for example, in the case of $\mathfrak{f}_4$ there are no other involutive automorphisms).

The Root Method gives the description of a compact simple Lie algebra by the type of root system which is a rather complicated invariant of a Lie algebra.

Mirror Geometry gives the description of a compact simple Lie algebra by the type of principal unitary involutive automorphism being a simple algebraic-geometric characteristic of a Lie algebra.

The Root Method does not give a classification of simple compact Lie algebras in the sense of the Invariant Theory, that is, does not give the method of construction of a canonical base: there the problem of classification is solved by the 'guessing' of a concrete Lie algebra with an admissible root system.

Mirror Geometry is, in essence, the method of determination of a canonical base in a Lie algebra.

The problems in applications to tri-symmetric spaces of rank 1, for example, can be solved in the 'Root Method' by the observation of all possibilities of the list of E. Cartan. Mirror Geometry gives a direct approach to symmetric spaces of rank 1, avoiding the general classification. Moreover, any theorem of Mirror Geometry is a theorem of the theory of symmetric spaces (after some trivial reformulation). This is not valid for the Root Method.

The Root Method is not effective in the theory of homogeneous Riemannian spaces with mirrors (that is, all cases of homogeneous Riemannian spaces with $ds^2 > 0$ and non-trivial isotropy group). Mirror Geometry gives in this case the system of standard mirrors.

Of course, there are some problems when the possibilities of the Root Method are obviously effective but the possibilities of Mirror Geometry are not yet evident enough. Perhaps, here we need more systematic development in the future.

One may ask whether Mirror Geometry can be obtained from the Root Method. The simple example of an iso-involutive sum of index 1 and of type 1 for a simple compact Lie algebra demonstrates that Mirror Geometry and the Root Method are in some sense opposite. Indeed, the constructions of the Root Method depend on 'regular vectors', whereas in the above example the conjugating automorphism is generated by a singular vector.