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Analytic Functionals on the Sphere

Mitsuo Morimoto

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ABSTRACT. This book treats, first of all, spherical harmonic expansion of real-analytic functions and hyperfunctions on the (real) sphere. To study this, we construct a system of good complex neighborhoods of the sphere in the complex sphere by means of the Lie norm, and consider holomorphic functions and analytic functionals on the complex neighborhoods. The book then treats harmonic functions on the Euclidean ball and complex harmonic functions on the Lie ball. It also discusses the Fourier-Borel transformation of analytic functionals on the complex sphere. This English edition is a further development of the author's lecture notes circulated in Japanese.

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(Continued in the back of this publication)

Analytic Functionals on the Sphere

Preface

In his first paper [**35**] on the theory of hyperfunctions, M. Sato treated the Fourier expansion of hyperfunctions on the unit circle in connection with the Laurent expansion at the origin of the complex plane.

The characterization of spherical harmonic expansion of hyperfunctions on the sphere $\mathbb{S}^n = \{x \in \mathbb{R}^{n+1}; x^2 = x_1^2 + x_2^2 + \cdots + x_{n+1}^2 = 1\}$ was first obtained by Hashizume-Minemura-Okamoto [**7**]. Their methods rely on the characterization of real analytic functions by means of the Laplace-Beltrami operator (see Lions-Magenes [**16**] and Seeley [**37**]) and can be applied not only to the sphere but also to a general compact real analytic manifold. But their methods are far from the complex analysis that Sato employed in the case of the one-dimensional sphere, that is, the circle.

In this book, we shall construct a complex neighborhood $\tilde{\mathbb{S}}^n(r)$ of the sphere $\mathbb{S}^n$ by means of the Lie norm. The complex neighborhood $\tilde{\mathbb{S}}^n(r)$ is a direct generalization of the annular neighborhood $\{z \in \mathbb{C}; \frac{1}{r} < |z| < r\}$ of the circle and allows us a complex analysis approach to the theory of spherical harmonic expansion.

Let us overview the monograph chapter by chapter.

In Chapter 1, we recall, as a motivation for later chapters, some facts on Fourier expansion of real analytic functions, C^∞ functions, distributions, and hyperfunctions on the circle. These materials are well-known to a specialist but students may find them useful.

Our first tool for studying the higher-dimensional sphere is, of course, the classical theory of spherical harmonics. Following mainly Müller [**33**] and referring also to Vilenkin [**43**] and to Stein-Weiss [**41**], we present it in Chapter 2 with detailed calculation. (Recently another related book was published by Müller [**34**].) We also state the characterization of C^∞ functions and distributions on $\mathbb{S}^n$ by the growth condition of their spherical harmonic expansion. In §2.8 and §2.9 we study the Poisson formula which represents the unique harmonic function in the unit ball having a given continuous boundary value on the sphere $\mathbb{S}^n$.

Our second tool is the cross norm, which is presented in the first two sections of Chapter 3 according to Drużkowski [**1**]. The Lie norm $L(z)$ on $\mathbb{C}^{n+1}$ is the cross norm of the Euclidean norm on $\mathbb{R}^{n+1}$ and is powerful enough to estimate spherical harmonics.

In the later sections of Chapter 3, we introduce the Lie ball $\tilde{B}(r) = \{z \in \mathbb{C}^{n+1}; L(z) < r\}$, which turns out to be E. Cartan's classical bounded domain of type 4 (see Hua [**12**]). We study the space of holomorphic functions on $\tilde{B}(r)$ and their expansion by homogeneous polynomials. The Shilov boundary of $\tilde{B}(r)$ is called the Lie sphere. (See [**20**] for the spherical harmonic expansion of hyperfunctions on the Lie sphere.) The complex sphere $\tilde{\mathbb{S}}^n = \{z \in \mathbb{C}^{n+1}; z^2 = z_1^2 + z_2^2 + \cdots + z_{n+1}^2 = 1\}$ is the natural complexification of the sphere $\mathbb{S}^n$. We put $\tilde{\mathbb{S}}^n(r) = \tilde{\mathbb{S}}^n \cap \tilde{B}(r)$. The family $\{\tilde{\mathbb{S}}^n(r); r > 1\}$ is a fundamental system of complex neighborhoods of the sphere $\mathbb{S}^n$. We shall find a characterization of holomorphic functions on $\tilde{\mathbb{S}}^n(r)$ by the growth condition of their spherical harmonic expansion.

In Chapter 4, we introduce hyperfunctions on $\mathbb{S}^n$ and more generally analytic functionals on the complex sphere $\tilde{\mathbb{S}}^n$. Then in §4.10, we show that a harmonic function in the unit ball is in one-to-one correspondence with its hyperfunction boundary value on $\mathbb{S}^n$. This fact motivated the introduction of hyperfunctions.

A special case of the complex sphere is the complex light cone $\tilde{\mathbb{S}}_0 = \{z \in \mathbb{C}^{n+1}; z^2 = z_1^2 + z_2^2 + \cdots + z_{n+1}^2 = 0\}$. Because of the cone structure, we can develop the theory of expansion holomorphic functions and analytic functionals on $\tilde{\mathbb{S}}_0$ into homogeneous components. It was our starting point (see Morimoto-Fujita [**27**]) but in this book we try to state the theory in the complex sphere of complex radius in general.

Now suppose a hyperfunction T on the sphere $\mathbb{S}^n$ is given. If we define the function $F(\xi)$ on $\mathbb{R}^{n+1}$ by

$$F(\xi) = \langle T_\omega, \exp(i\lambda\xi \cdot \omega)\rangle \tag{0.1}$$

the function $F(\xi)$ satisfies the differential equation

$$(\Delta_\xi + \lambda^2)F(\xi) = 0. \tag{0.2}$$

But a solution of the differential equation (0.2) is not always represented in the form of (0.1) with a hyperfunction T. In order to represent all the solutions of (0.2), T in the formula (0.1) should be something more general than hyperfunctions (see Hashizume-Kowata-Minemura-Okamoto [**6**]). In the case of $n = 1$, Helgason [**8**] showed that this "something" is an analytic functional of a certain kind (entire functional). We obtained some detailed results in the case of $n = 1$ in [**19**], and then, in [**22**], proved that this "something" is an entire functional for general n; we treat this topic in Chapter 5 (see also Helgason [**10**]).

In Chapter 6, we introduce several spaces of entire functions, which will be used to describe the image of the Fourier-Borel transformation in subsequent chapters.

Chapter 7 is based on Morimoto-Fujita [**31**], where we introduce the Fourier-Borel transformation for analytic functionals on the complex sphere $\tilde{\mathbb{S}}_\lambda$. The image turns out to be the space of λ-harmonic entire functions.

Chapter 8 is based on Morimoto-Fujita [**30**], where we introduce the spherical Fourier-Borel transformation of γ-harmonic functionals on the Lie ball. The image turns out to be the space of entire functions on the complex sphere $\tilde{\mathbb{S}}_\gamma$.

The present volume is an enlarged version of my lecture note [**21**] on spherical harmonic expansion of hyperfunctions on the sphere, which was intended to complement the book [**17**] on hyperfunctions and microfunctions and the lecture note [**18**] on the Fourier transformation of hyperfunctions.

Chapter 1 was read at Rikkyo University from April to July, 1978. From September 1978 for one year of my sabbatical leave of absence, I stayed in Europe and attended seminars on harmonic analysis at the Universities of Nancy, of Strasbourg, and of Lyon, where I read some part of this lecture note.

In May, 1979, I was invited by Stephan Banach International Mathematical Center as a lecturer at the Semester on Complex Analysis and read [**22**], which was published 4 years later. The discussion with Professor J. Siciak at the semester was stimulating; I learned the Lie norm and could improve my results considerably.

The aim of [**21**] was to present a self-contained exposition of the theory of holomorphic functions and analytic functions on the complex sphere using the Lie norm. The lecture was read at Sophia University in 1980 and in 1987.

Meanwhile, many interesting results of the theory have been obtained by R. Wada [**45**], [**48**], and [**46**], especially in connection with the complex light cone. I tried to present them in a unified manner in this enlarged version, which was prepared while visiting the University of Maryland at College Park from November 1992 to March 1993 (see [**23**]).

After coming back to Japan, I was able to obtain several results jointly with K. Fujita [**3**], [**26**], [**27**], [**2**], [**28**], [**29**], [**30**], [**31**], [**4**], [**5**]. I try to incorporate them into this book as much as possible.

March 19, 1998
Mitsuo Morimoto
(Sophia University)

Contents

CHAPTER 1

Fourier expansion of hyperfunctions on the circle

In this chapter we study the Fourier expansion of functions and hyperfunctions on the unit circle, which is the one-dimensional sphere of radius 1.

In §1.1 we introduce various spaces of functions on the circle and in §1.2 recall well-known results on the Fourier series. We define distributions and hyperfunctions on the unit circle in §1.3 and consider the Fourier expansion of distributions in §1.4 and that of hyperfunctions in §1.5.

The Fourier expansion of a hyperfunction does not converge as a function. In §1.6 we introduce the Poisson integral of a hyperfunction, which can be considered as a method of summation of the Fourier series and will be extended to higher-dimensional cases in later chapters. The Fourier-Borel transform of a hyperfunction is also considered as a method of summation and is discussed in §1.7. In the last §1.8 we mention the Cauchy-Hilbert transform of a hyperfunction but we shall not treat this concept in this book for the higher-dimensional case.

1.1. Function spaces on the circle

Let K be a compact set in the complex plane $\mathbb{C}$. For $\varepsilon > 0$, we define the ε-neighborhood K_ε by $K_\varepsilon = K + \{z \in \mathbb{C}; |z| < \varepsilon\}$. We say that f is a *germ of holomorphic functions* on K if there is $\varepsilon > 0$ such that f is a holomorphic function defined on the open set K_ε. We denote by $\mathcal{O}(K)$ the space of germs of holomorphic functions on K. $\mathcal{O}(K)$ is a complex linear space.

We denote by $\mathcal{O}_b(K_\varepsilon)$ the space of continuous functions on $\overline{K}_\varepsilon$ that are holomorphic on K_ε. We endow $\mathcal{O}_b(K_\varepsilon)$ with the norm $\|f\|_{C(K_\varepsilon)} = \sup\{|f(z)|; z \in K_\varepsilon\}$.

PROPOSITION 1.1. *$\mathcal{O}_b(K_\varepsilon)$ is a Banach space (that is, complete normed space) equipped with the norm $\|f\|_{C(K_\varepsilon)}$.*

PROOF. It is clear that it is a normed space. Let us show the completeness. Assume that

$$\|f_k - f_\ell\|_{C(K_\varepsilon)} \to 0 \quad (k, \ell \to \infty).$$

For any $z \in \overline{K}_\varepsilon$, the sequence $\{f_k(z)\}$ is a Cauchy sequence. We denote by $f(z)$ its limit. As the sequence of functions f_k converges to the function f uniformly on $\overline{K}_\varepsilon$, f is continuous on $\overline{K}_\varepsilon$ and holomorphic in K_ε. That is, $f \in \mathcal{O}_b(K_\varepsilon)$ and $\|f - f_k\|_{C(K_\varepsilon)} \to 0 \quad (k \to \infty)$. □

We can reformulate the definition of $\mathcal{O}(K)$:

$$\mathcal{O}(K) = \text{ind lim}\{\mathcal{O}_b(K_\varepsilon); \varepsilon > 0\}.$$

More intuitively, we can say

$$\mathcal{O}(K) = \bigcup\{\mathcal{O}_b(K_\varepsilon); \varepsilon > 0\},$$

where two functions will be identified if they coincide in a neighborhood of K.

By the theory of the inductive limit of topological linear spaces, we have the following theorem:

THEOREM 1.2. *There is a unique Hausdorff locally convex linear topology on* $\mathcal{O}(K)$ *satisfying the following conditions* (i) *and* (ii) :

(i) *For any* $\varepsilon > 0$, *the mapping* $\rho_\varepsilon : \mathcal{O}_b(K_\varepsilon) \to \mathcal{O}(K)$ *is continuous*;

(ii) *Suppose* B *is a locally convex linear space and* $T : \mathcal{O}(K) \to B$ *is a linear mapping. If* $T \circ \rho_\varepsilon : \mathcal{O}_b(K_\varepsilon) \to B$ *is continuous for any* $\varepsilon > 0$, *then* T *is continuous.*

We omit the proof. We refer the reader, for example, to Komatsu [**15**].

We call the mapping $\rho_\varepsilon : \mathcal{O}_b(K_\varepsilon) \to \mathcal{O}(K)$ defined above the *restriction mapping.* In the sequel, we equip $\mathcal{O}(K)$ with this inductive limit topology.

We are interested in the case $K = \mathbb{S}^1$, where $\mathbb{S}^1 = \{z \in \mathbb{C}; |z| = 1\}$ is the unit circle. The space $\mathcal{A}(\mathbb{S}^1)$ of *real analytic functions* on the unit circle $\mathbb{S}^1$ is, by definition, the space $\mathcal{O}(\mathbb{S}^1)$.

A function $f(z)$ on $\mathbb{S}^1$ can be identified with the periodic function $\hat{f}$ of period 2π defined by $\hat{f}(t) = f(e^{it})$, $t \in \mathbb{R}$. Let $f \in \mathcal{O}(\mathbb{S}^1_\varepsilon)$. Consider $\hat{f}(\tau) = f(e^{i\tau})$, $\tau = t + is$. Since $|e^{i\tau}| = e^{-s}$, $\hat{f}(\tau)$ is holomorphic in the band

$$\{\tau = t + is; -\log(1+\varepsilon) < s < -\log(1-\varepsilon)\}$$

and periodic of period 2π. In particular, if $f \in \mathcal{A}(\mathbb{S}^1)$, then $\hat{f}(t)$ is a C^∞ function.

A function f on $\mathbb{S}^1$ is a C^∞ *function* if $\hat{f}(t)$ is a C^∞ periodic function of period 2π on $\mathbb{R}$. We denote by $\mathcal{E}(\mathbb{S}^1)$ the space of C^∞ functions on $\mathbb{S}^1$ and define the seminorms

$$\|f\|_j = \sup\left\{|\hat{f}^{(j)}(t)|; 0 \le t < 2\pi\right\}$$

on it. We equip $\mathcal{E}(\mathbb{S}^1)$ with the topology defined by the system of seminorms $\|f\|_j$, $j = 0, 1, 2, \cdots$.

PROPOSITION 1.3. $\mathcal{E}(\mathbb{S}^1)$ *is a Fréchet space* (*locally convex complete metrizable space*).

PROOF. Since the topology of $\mathcal{E}(\mathbb{S}^1)$ is defined by a countable system of seminorms, it is metrizable. Let us show the completeness. Suppose

$$\|f_k - f_\ell\|_j \to 0 \quad (k, \ell \to \infty)$$

for any j. We denote by $\hat{g}_j(t)$ the limit function of the Cauchy sequence $\hat{f}_k^{(j)}(t)$. The functions $\hat{g}_j(t)$ are continuous and the sequence $\hat{f}_k^{(j)}$ converges uniformly to $\hat{g}_j$ as $k \to \infty$ for any j. Then the following lemma asserts that $\hat{g}_0$ is C^j and $\hat{g}_j = \hat{g}_0^{(j)}$ for any j. This proves the proposition. □

LEMMA 1.4. *Suppose* $\hat{f}_k$ *are* C^1 *and* $\hat{f}_k(t) \to \hat{g}(t)$ $(k \to \infty)$ *at every point* t. *Suppose also* $\hat{f}'_k(t)$ *converges to* $\hat{g}_1(t)$ *uniformly. Then* $\hat{g}$ *is* C^1 *and we have* $\hat{g}'(t) = \hat{g}_1(t)$.

For the proof, we refer the reader, for example, to Takagi [**38**].

THEOREM 1.5. *The inclusion mapping* $\mathcal{A}(\mathbb{S}^1) \hookrightarrow \mathcal{E}(\mathbb{S}^1)$ *is continuous.*

PROOF. By Theorem 1.2, we have to show that the mapping $\mathcal{O}_b(\mathbb{S}^1_\varepsilon) \to \mathcal{E}(\mathbb{S}^1)$ is continuous for any fixed $\varepsilon > 0$. If $f \in \mathcal{O}_b(\mathbb{S}^1_\varepsilon)$, then we have

$$f^{(j)}(z) = \frac{j!}{2\pi i}\int_{|w-z|=\varepsilon} \frac{f(w)}{(w-z)^{j+1}}dw$$

for $z \in \mathbb{S}^1$. Therefore, we have

$$|f^{(j)}(z)| \leq \frac{j!}{2\pi}\frac{1}{\varepsilon^{j+1}}\|f\|_{\mathbb{S}^1_\varepsilon} 2\pi\varepsilon = j!\varepsilon^{-j}\|f\|_{\mathbb{S}^1_\varepsilon}.$$

Because $\hat{f}(t) = f(e^{it})$, we have

$$\begin{aligned}
&\hat{f}'(t) = ie^{it}f'(e^{it}),\\
&\hat{f}''(t) = -e^{2it}f''(e^{it}) - e^{it}f'(e^{it}),\\
&\hat{f}'''(t) = -ie^{3it}f'''(e^{it}) - 3ie^{2it}f''(e^{it}) - ie^{it}f'(e^{it})\\
&\cdots
\end{aligned}$$

and $\hat{f}^{(j)}$ can be represented as a linear combination of

$$e^{it}f'(e^{it}),\ e^{2it}f''(e^{it}),\ \cdots, e^{jit}f^{(j)}(e^{it}).$$

Therefore, there is a constant $C_j \geq 0$ such that $\|f\|_j \leq C_j\|f\|_{C(\mathbb{S}^1_\varepsilon)}$ for any $f \in \mathcal{O}_b(\mathbb{S}^1_\varepsilon)$. This means that $\mathcal{O}_b(\mathbb{S}^1_\varepsilon) \to \mathcal{E}(\mathbb{S}^1)$ is continuous. □

1.2. Fourier series

We recall the classical theory of Fourier series.

Let g be a continuous periodic function of period 2π. We wish to express $g(t)$ as follows:

$$g(t) = \sum_{k=-\infty}^{\infty} A_k e^{ikt}, \quad A_k \in \mathbb{C}. \tag{1.1}$$

If the right-hand side of (1.1) converges uniformly, then we have

$$\int_0^{2\pi} g(t)e^{-\ell t}dt = \sum_{k=-\infty}^{\infty} A_k \int_0^{2\pi} e^{i(k-\ell)t}dt = 2\pi A_\ell$$

and hence

$$A_k = \frac{1}{2\pi}\int_0^{2\pi} g(t)e^{-ikt}dt, \quad k = 0, \pm 1, \pm 2, \cdots. \tag{1.2}$$

Suppose now both $g(t)$ and $h(t)$ are continuous periodic functions of period 2π. We define a sesquilinear form $(g,h)_{\mathbb{S}^1}$ by

$$(g,h)_{\mathbb{S}^1} = \frac{1}{2\pi}\int_0^{2\pi} g(t)\overline{h(t)}dt.$$

By this notation, we can rewrite (1.2) as $A_k = (g(t), e^{ikt})_{\mathbb{S}^1}$.

For a periodic function g of period 2π, we define the Fourier coefficient $\tilde{g}(k)$ of $g(t)$ by $\tilde{g}(k) = (g(t), e^{ikt})_{\mathbb{S}^1}$ and call the formal series

$$\sum_{k=-\infty}^{\infty} \tilde{g}(k)e^{ikt} \tag{1.3}$$

the *Fourier series* of $g(t)$ and

$$S_N(t) = \sum_{k=-N}^{N} \tilde{g}(k)e^{ikt} \tag{1.4}$$

the N-th *Fourier partial sum* of $g(t)$.

REMARK 1.6. There is a continuous function for which the Fourier series (1.3) does not converge at any point t. R. T. Seeley [**36**] is a good introduction to Fourier series and the reader can find there a list of literature.

Despite this remark, we can modify the meaning of convergence in such a way that the Fourier series (1.3) converges and that we have

$$g(t) = \lim_{N\to\infty} S_N(t) = \sum_{k=-\infty}^{\infty} \tilde{g}(k)e^{ikt}. \tag{1.5}$$

Here we mention three methods: the Poisson method, the Féjer method, and the L^2 method.

THEOREM 1.7 (Poisson). *Let g be a continuous periodic function of period 2π. For any r with $0 \le r < 1$,*

$$u(r,t) = \sum_{k=-\infty}^{\infty} \tilde{g}(k)r^{|k|}e^{ikt} \tag{1.6}$$

converges, where the convergence is uniform in $0 \le t \le 2\pi$, $0 \le r \le 1-\varepsilon$ for any fixed $\varepsilon > 0$. The function $u(r,t)$ defined by (1.6) is a harmonic function on the open unit disk; that is,

$$\Delta u = 0,$$

where

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial t^2}, \quad x = r\cos t,\ y = r\sin t, \tag{1.7}$$

is the Laplacian. *Further, we have*

$$\lim_{r\to 1-0} u(r,t) = g(t), \tag{1.8}$$

where the convergence is uniform.

Sometimes this method is called the *Abel summation method.*

PROOF. Since g is continuous, $M = \sup\{|g(t)|; 0 \le t \le 2\pi\} < \infty$ and

$$|\tilde{g}(k)| \le \frac{1}{2\pi}\int_0^{2\pi} |g(t)|dt \le M.$$

Thus, $\{\tilde{g}(k)\}$ is a bounded sequence of numbers. Therefore, comparing with the geometric series, we can show that (1.6) converges uniformly. Because every term in (1.6) is harmonic, the uniform limit function u is also harmonic.

On the other hand, by a simple calculation, the function $u(r,t)$ is given by the *Poisson integral* of g:

$$u(r,t) = \frac{1}{2\pi}\int_0^{2\pi} \frac{1-r^2}{1-2r\cos(s-t)+r^2} g(s)ds.$$

From this formula we can show (1.8). (We omit the details.) □

The following Féjer theorem is also well-known. For the details, we refer the reader, for example, to Takagi [**38**].

THEOREM 1.8 (Féjer). *Let g be a continuous periodic function of period 2π and $S_N(t)$ be the N-th Fourier partial sum* (1.4) *of $g(t)$. Then we have*

$$\lim_{N\to\infty}\frac{1}{N}(S_1(t)+S_2(t)+\cdots+S_N(t))=g(t), \tag{1.9}$$

where the convergence is uniform.

(1.9) is sometimes called the *Cesàro summation method.*

PROOF. We use the formula

$$\frac{1}{N}(S_1(t)+S_2(t)+\cdots+S_N(t))=\frac{1}{2\pi N}\int_0^{2\pi}g(s)\left[\frac{\sin\dfrac{N(t-s)}{2}}{\sin\dfrac{(t-s)}{2}}\right]^2 ds.$$

(The details are omitted.) □

Finally, we mention the L^2 method.

THEOREM 1.9. *Let g be a continuous periodic function of period 2π. (g may be a square integrable function.) Then we have*

$$\sum_{k=-\infty}^{\infty}|\tilde{g}(k)|^2=\|g\|^2_{\mathbb{S}^1}\equiv\frac{1}{2\pi}\int_0^{2\pi}|g(t)|^2dt$$

and

$$\lim_{N\to\infty}\|S_N-g\|_{\mathbb{S}^1}=0.$$

Theorem 1.9 is one of the first theorems in the theory of Hilbert spaces. We omit the details.

It is easy to prove the following lemma.

LEMMA 1.10. *Let $f_1, f_2, \cdots$ be a sequence of continuous functions defined on $[0, 2\pi]$ and put $s_N = f_1 + f_2 + \cdots + f_N$. If $s_N(t)$ converges uniformly to $f(t)$ as $N \to \infty$, then*

(i) $\sum_{k=1}^{\infty} r^k f_k(t) \to f(t)$ *uniformly as* $r \to 1-0$,
(ii) $\frac{1}{N}(s_1 + s_2 + \cdots + s_N) \to f(t)$ *uniformly as* $N \to \infty$,
(iii) $\|s_N - f\|_{\mathbb{S}^1} \to 0$ *as* $N \to \infty$.

As remarked earlier, the converse is not always true. (We have a counter-example in the Fourier series.) If we assume smoothness on g, then we can prove the uniform convergence of the Fourier series.

THEOREM 1.11. *Let g be a C^1 periodic function of period 2π. Then the N-th Fourier partial sum $S_N(t)$ of $g(t)$ converges to $g(t)$ uniformly as $N \to \infty$.*

PROOF. We put

$$A'_k=\frac{1}{2\pi}\int_0^{2\pi}g'(t)e^{-ikt}dt.$$

If $k \neq 0$, then by integration by parts we have

$$\tilde{g}(k)=\frac{1}{2\pi}\int_0^{2\pi}g(t)e^{-ikt}dt=\frac{1}{ik}A'_k.$$

Therefore, by Theorem 1.9, we get

$$\sum_{k\neq 0}|\tilde{g}(k)| = \sum_{k\neq 0}\frac{1}{k}|A'_k| \leq \sqrt{\sum_{k\neq 0}\frac{1}{k^2}}\sqrt{\sum_{k\neq 0}|A'_k|^2} < \infty.$$

This shows that S_N converges uniformly as $N \to \infty$. The limit function is equal to $g(t)$ because of Lemma 1.10 and Theorem 1.9. □

DEFINITION 1.12. A sequence of numbers $\{A_k\}_{k\in\mathbb{Z}}$ is *rapidly decreasing* if

$$\sup\left\{|k|^j|A_k|;\, k \in \mathbb{Z}\right\} < \infty \text{ for any } j \geq 0.$$

For the Fourier expansion of C^∞ functions we have the following theorem:

THEOREM 1.13. *Let g be a C^∞ periodic function of period 2π. Then the sequence of Fourier coefficients $\{\tilde{g}(k)\}_{k\in\mathbb{Z}}$ is rapidly decreasing and we have* (1.5) *in the topology of $\mathcal{E}(\mathbb{S}^1)$.*

Conversely, let $\{A_k\}_{k\in\mathbb{Z}}$ be a rapidly decreasing sequence. Then

$$g(t) = \sum_{k=-\infty}^{\infty} A_k e^{ikt} \tag{1.10}$$

converges in the topology of $\mathcal{E}(\mathbb{S}^1)$ and satisfies $\tilde{g}(k) = A_k$.

PROOF. By integration by parts, the Fourier coefficient of $g^{(j)}(t)$ is given by

$$\begin{aligned}
\widetilde{g^{(j)}}(k) &= \frac{1}{2\pi}\int_0^{2\pi} g^{(j)}(t)e^{-ikt}dt \\
&= \frac{1}{2\pi}\left\{[g^{(j-1)}(t)e^{-ikt}]_0^{2\pi} + \int_0^{2\pi} g^{(j-1)}(t)ike^{-ikt}dt\right\} \\
&= (ik)\tilde{g}^{(j-1)}(k) = \cdots = (ik)^j\tilde{g}(k).
\end{aligned}$$

Since $\{\widetilde{g^{(j)}}(k)\}$ is bounded for any j, $\{\tilde{g}(k)\}$ is a rapidly decreasing sequence. The Fourier expansion of $g^{(j)}$ is given by

$$g^{(j)}(t) = \sum_{k=-\infty}^{\infty} (ik)^j\tilde{g}(k)e^{ikt},$$

where the convergence is uniform by Theorem 1.11. Because

$$S_N^{(j)}(t) = \sum_{k=-N}^{N} (ik)^j\tilde{g}(k)e^{ikt},$$

we have $S_N(t) \to g(t)$ in the topology of $\mathcal{E}(\mathbb{S}^1)$ as $N \to \infty$.

Conversely, if a sequence $\{A_k\}_{k\in\mathbb{Z}}$ is rapidly decreasing, then Lemma 1.4 implies that (1.10) is a C^∞ function and $\tilde{g}(k) = A_k$. □

Because $S_N(t)$ is a real analytic function, we have the following corollary.

COROLLARY 1.14. *$\mathcal{A}(\mathbb{S}^1)$ is dense in $\mathcal{E}(\mathbb{S}^1)$.*

1.3. Distributions and hyperfunctions

A *hyperfunction* on the unit circle $\mathbb{S}^1$ is a continuous linear functional on $\mathcal{A}(\mathbb{S}^1)$ and a *distribution* on $\mathbb{S}^1$ is a continuous linear functional on $\mathcal{E}(\mathbb{S}^1)$. We denote by $\mathcal{B}(\mathbb{S}^1)$ the space of hyperfunctions on $\mathbb{S}^1$ and by $\mathcal{E}'(\mathbb{S}^1)$ the space of distributions on $\mathbb{S}^1$.

The canonical bilinear form of duality on $\mathcal{B}(\mathbb{S}^1) \times \mathcal{A}(\mathbb{S}^1)$ is denoted by $\langle T, g\rangle$. We denote also by the same notation $\langle T, g\rangle$ the canonical bilinear form on $\mathcal{E}'(\mathbb{S}^1) \times \mathcal{E}(\mathbb{S}^1)$.

The convergence in the topology of $\mathcal{B}(\mathbb{S}^1)$ is defined weakly; we say a sequence $\{T_j\}$ of hyperfunctions converges to a hyperfunction T in the topology of $\mathcal{B}(\mathbb{S}^1)$ if and only if

$$\langle T_j, g\rangle \to \langle T, g\rangle \text{ for any } g \in \mathcal{A}(\mathbb{S}^1).$$

The convergence in the topology of $\mathcal{E}'(\mathbb{S}^1)$ is also defined weakly.

If f is an integrable function on $\mathbb{S}^1$, then we can define $T_f \in \mathcal{E}'(\mathbb{S}^1)$ by

$$T_f : \mathcal{E}(\mathbb{S}^1) \ni g \mapsto (g, f)_{\mathbb{S}^1} \equiv \frac{1}{2\pi}\int_0^{2\pi} g(t)\overline{f(t)}dt.$$

The mapping $f \mapsto T_f$ is an antilinear injection. (For the proof of this fact, we should go back to the definition of Lebesgue integration.) In the sequel, we shall identify the distribution T_f with the integrable function f.

THEOREM 1.15. *A distribution on $\mathbb{S}^1$ is a hyperfunction; that is, there is a linear injection*

$$\mathcal{E}'(\mathbb{S}^1) \hookrightarrow \mathcal{B}(\mathbb{S}^1). \tag{1.11}$$

PROOF. The mapping $\mathcal{A}(\mathbb{S}^1) \hookrightarrow \mathcal{E}(\mathbb{S}^1)$ is continuous (Theorem 1.5). Restricting $T \in \mathcal{E}'(\mathbb{S}^1)$ to $\mathcal{A}(\mathbb{S}^1)$, we can define the mapping (1.11). It is injective since $\mathcal{A}(\mathbb{S}^1)$ is dense in $\mathcal{E}(\mathbb{S}^1)$ (Corollary 1.14). □

1.4. Fourier expansion of distributions

DEFINITION 1.16. A sequence of numbers $\{B_k\}_{k\in\mathbb{Z}}$ is *slowly increasing* if there exist $C \geq 0$ and $j \geq 0$ such that

$$|B_k| \leq C(1 + |k|^j), \quad k \in \mathbb{Z}.$$

For simplicity, sometimes we omit the subscript and denote $\{B_k\}_{k\in\mathbb{Z}}$ by $\{B_k\}$.

For $T \in \mathcal{E}'(\mathbb{S}^1)$, we put

$$\tilde{T}(k) = \overline{\langle T(t), e^{ikt}\rangle}, \quad k \in \mathbb{Z}$$

and call them the *Fourier coefficients* of T.

THEOREM 1.17. *Suppose $T \in \mathcal{E}'(\mathbb{S}^1)$. Then the sequence of Fourier coefficients $\{\tilde{T}(k)\}$ is slowly increasing on $\mathbb{Z}$ and we have*

$$T(t) = \sum_{k=-\infty}^{\infty} \tilde{T}(k)e^{ikt} \tag{1.12}$$

in the topology of $\mathcal{E}'(\mathbb{S}^1)$. For $T \in \mathcal{E}'(\mathbb{S}^1)$ and $g \in \mathcal{E}(\mathbb{S}^1)$, we have

$$\langle T, g\rangle = \sum_{k=-\infty}^{\infty} \overline{\tilde{T}(k)}\tilde{g}(k).$$