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Guest Editor:
B.H.V. Topping

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Editorial

This sixth volume of *Computational Technology Reviews* includes a selection of papers which were originally presented as invited review lectures at *The Eleventh International Conference on Computational Structures Technology* (CST 2012) and *The Eighth International Conference on Engineering Computational Technology* (ECT 2012) held concurrently in Dubrovnik, Croatia from 4-7 September 2012. I am grateful to the authors and co-authors of the papers included in this volume. Their contribution to these conferences and to this volume of *Computational Technology Reviews* is greatly appreciated.

Other papers presented at these conferences are published as follows:

- *Other Invited Review Lectures from CST 2012 and ECT 2012 are published in:* Computational Technology Reviews, Volume 5, Saxe-Coburg Publications, Stirlingshire, Scotland, 2012.
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Contents

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Recent Developments in the Integration of Computer Aided Design and Analysis

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Abstract

For linear elastic problems, it is well-known that mesh generation dominates the total analysis time. Different types of methods have been proposed to directly or indirectly alleviate this burden associated with mesh generation. This paper reviews a subset of such methods centred on tighter coupling between computer aided design (CAD) and analysis (finite element or boundary element methods). It focuses specifically on frameworks which rely on constructing a discretisation directly from the functions used to describe the geometry of the object in CAD. Examples include B-spline subdivision surfaces, isogeometric analysis, NURBS-enhanced FEM and parametric-based implicit boundary definitions. Recent advances in these methods are reviewed and compared with other paradigms which also aim at alleviating the burden of mesh generation in computational mechanics.

Keywords: isogeometric analysis, review, mesh burden reduction, isogeometric boundary element methods, NURBS-enhanced finite element method.

1 Introduction

The finite element method is the most widely used numerical method in practice and is underlined by fifty years of rigorous mathematical analysis. It offers results whose quality can be predicted *a priori* and controlled *a posteriori* for a range of problems of engineering relevance. The finite element method relies on the creation of a data structure known as "mesh" which is used to construct the approximation (of both the geometry of the domain and the field variables) and to perform numerical integration. The mesh is a set of elements of polyhedral shape (with straight or curved edges or faces) covering the domain. Traditionally, these elements have been simplex shapes (triangles and tetrahedra) or of quadrangles or hexahedral shapes, but recently

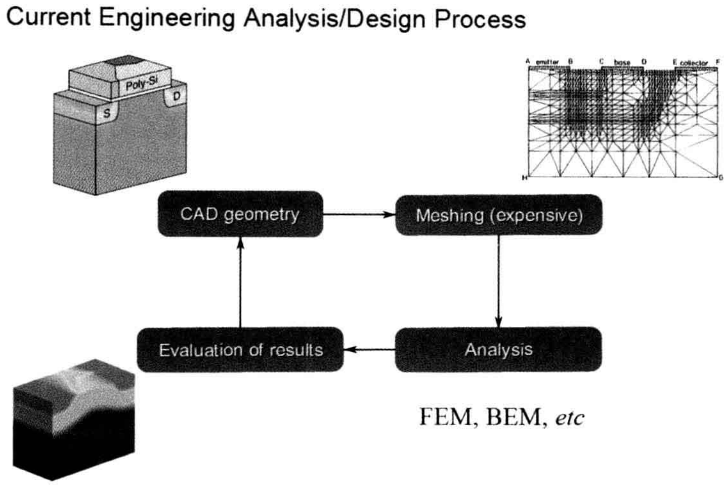

Figure 1: Iterative design process. It is evident that such an iterative process is cumbersome and time consuming because a new, analysis-suitable, mesh must be generated for each new geometry. Several ideas have been advanced in the literature to try and decrease the mesh generation burden

developed elements [140] and numerical integration techniques [103] allow the development of elements with an arbitrary number of edges/faces.

For day to day industrial problems, initial analyses during early stage design are usually carried out using the assumption of linear elasticity. The geometry of the domain is most commonly generated using CAD software by a designer or engineer. A mesh must then be created to approximate this geometry and discretise the partial differential equation (PDE) governing the problem to allow analysis to be performed. The elements must fulfil a list of quality criteria related to their shape and grading throughout the domain, and this step is still difficult and far from being automated. Once the analysis results are available, it is typically necessary for the analyst to go back to the designer and propose changes to the design so that the part meets certain criteria related to, *e.g.* the maximum allowable stress or the maximum deflection in the component. In turn, a new mesh must be generated to approximate the new geometry and perform a new analysis. This iterative process must be performed as many times as necessary to converge to a suitable geometry. The idea is illustrated in Figure 1.

It is evident that such an iterative process is cumbersome and time consuming because a new, analysis-suitable mesh must be generated for each new geometry, which is not a simple task. Indeed, meshes of good quality elements still today are difficult to generate automatically, or are restricted to simplex (tetrahedral or triangular) elements, which are usually not robust (too stiff and not amenable to incompressible materials). Moreover, it may happen in industrial practice that the geometry is so complex that available mesh generators fail, or require significant human intervention. This has spurred a vast amount of research to enable analysts to generate quality meshes quickly and robustly. An exciting avenue of investigation, followed by a large number of strong research groups worldwide consists of developing robust, fast, and automatic mesh generators, in particular for hexahedral elements. Some of the most notable work performed in this area can be non-exhaustively summarised as

- **Triangular or Tethedral-mesh generators:** Tetrahedral mesh generators are more flexible and robust for complex geometries compared to hexahedral mesh generators. The current meshing techniques can be roughly grouped into the following categories: advancing front [88, 89], octree-based methods [158], Delaunay approaches [81, 151] and mesh optimization techniques [44, 63]. Some of the latest developments in this area include works on a remeshing approach exploiting the natural anisotropy of most surfaces and the construction of high-quality isotropic tetrahedral meshes by using a variational principle [3, 4]. The reader is referred to the review paper of Alliez *et al.* [5] for details. The recent paper by Young *et al.* [159] is particularly noteworthy as it is on the generation of meshes from medical images (image-based mesh generation).
- **Quadrilateral or Hexahedral-mesh generators:** Hexahedral meshes are preferred from a numerical analysis perspective. However, it is more difficult to generate, as explained in the survey by Shepherd and Johnson [135]. Hexahedral meshes can be generated directly, or converted from the tetrahedral meshes indirectly. Zhu [162] is one of the first ones using an advancing front approach. Blacker and Stephenson [28] introduced a paving algorithm to provide a robust and efficient generator for all-quadrilateral meshes. Plastering is an extension of the paving algorithm for hexahedral mesh in three dimensions. The reader is referred to the survey in 1998 by Owen [113] for unstructured meshes and refinement. Recent developments include the work of Zhang and Bajaj [160] who proposed an isosurface extraction method to automatically extract adaptive and quality hexahedral meshes from volume data, Ito *et al.* [73] who introduced an octree-based mesh generation method to create geometry-adapted unstructured hexahedral meshes automatically from triangulated surface models, Staten *et al.* [138] who developed an new algorithm to leverage the benefits of paving and plastering, and Sarrate and Huerta [122] who proposed an efficient algorithm for complex geometries based on a recursive domain decomposition.

In parallel with these advances in meshing, several paradigms have been advanced in the literature to try and decrease the reliance of numerical analysis on a mesh. The ideas behind these hinge upon the following goals also summarised in Table 1:

- **Avoid altogether the generation of a mesh:** Meshfree methods (see Nguyen [104] for a recent review and computer implementation details including an open source MATLAB code) relax the notion of neighbour relationships (connectivity) by simplifying the addition, suppression and movement of points or

Goals	Meshfree methods	IGA	Element technology	Implicit boundary methods	Boundary discretision methods
high order continuity	Yes	Yes	No	No	No
alleviate mesh distortion	Yes	Yes	Yes	No	N/A
represent geometries exactly	No	Yes	No	No	No

Table 1: Several frameworks have been advanced in the literature to try and decrease the mesh generation burden.

nodes. These methods came about in the seventies with the smoothed particle hydrodynamics (SPH) method proposed by Gingold and Monaghan [65] and the element-free Galerkin method by Belytschko *et al.* [24]. They are not as well understood mathematically as the finite element method, and most methods rely on the definition of a "mesh", be it for numerical integration or generation of the point cloud. A rigorous quantification of the approximation property of meshfree methods, especially with reference to the influence of the point distribution is an open problem, although some headway has been made, *inter alia*, in the area of point collocation methods developed by Davydov and Oanh [47] and maximum entropy interpolants by Rosolen *et al.* [120].

- **Element technology:** Meshes of tetrahedral elements are much more easily generated than meshes of hexahedral elements (bricks) especially if user intervention is to be minimised. If robust and accurate tetrahedral elements could be formulated for most problems of engineering relevance, such fast mesh generators would become all the more useful. Several ideas have been proposed in the literature to avoid most of the limitations of (linear) tetrahedral elements. The node-averaged tetrahedral elements of Bonet and Burton [29], the locking free tetrahedral element of De Micheli and Mocellin [54] and the (cell, edge/face, node-based) smoothed finite element method (SFEM) of Liu *et al.* [85] are examples of such attempts. Although these methods have other drawbacks associated with complications for multi-material problems and larger band widths, the edge-based smoothed finite element method was shown to yield superior results to that of the standard FEM for problems in linear elasticity [85], visco-elastoplasticity [109] and geometrically nonlinear problems [108]. Polygonal or polyhedral elements are another alternative to ease the difficulties associated with generating quality meshes made up of robust elements. They can serve as transition elements during mesh generation, and therefore offer added flexibility. Work in this area includes the early work of Alwood and Cornes [6], Sukumar and Tabarraei [140], Euler *et al.* [61], Sukumar and Malsch [139], Dai *et al.* [46], Mousavi *et al.* [101] and Natarajan *et al.* [103].

- **Use the geometry data provided by CAD directly in the analysis:** From the description of the early-stage design process above, it is clear that the geometrical information provided by CAD is lost during the mesh generation process, because the smooth, arbitrary surfaces representing the boundary of the object are fitted as closely as possible during the mesh generation by piecewise polynomial surfaces (usually piecewise linear or quadratic) or the faces of the finite elements. The idea of the subdivision surfaces B-spline finite elements of Cirak *et al.* [37, 38, 39] and the isogeometric analysis of Hughes [70] is to use the geometrical description of the component to approximate the field variables. More precisely, the (usually Non-Uniform Rational B-Spline – NURBS) description of the boundary of the domain is used to build shape functions for the approximation of the unknown fields on the domain. In this way, any modification of the geometry of the object is immediately translated into a modified approximation over the new geometry, without needing to reconstruct a polynomial fit of the boundary. The first set of such methods were developed in the context of the finite element method [70], where a domain mesh is required to construct the NURBS-shape functions, and, more recently, for the boundary element method (BEM) in the paper by Politis *et al.* [116] and Simpson *et al.* [136], where mesh generation is completely avoided, bearing in mind the intrinsic limitations of the BEM. Advantages of the "isogeometric approach" include the use of the *exact geometry at all stages of the analysis* and, more significantly, a significant reduction or circumvention of mesh generation. But recent research also illustrated several other advantages and difficulties of the IGA approach. It is the goal of the present paper to give an overview of some of these developments. Recently, the work of Moumnassi *et al.* [100], based on the initial idea of Moës *et al.* [99] and Belytschko [25] showed that it was possible, using a marching method, to obtain an implicit domain definition based on multiple level set functions from arbitrary parametric surfaces provided from CAD data. This was a significant advance compared to previous work, as it enabled the treatment of corners and sharp edges (at the cost of a multiple-level set formulation). In some sense, this class of ambient space methods can be thought of as particular cases of the techniques described in the next paragraph.

- **Use a structured mesh independent of the geometry:** An important difficulty in the mesh generation process emanates from the requirement of the mesh to conform to the (usually complex) geometry of the domain. An alternative is to allow the geometry of the boundary of the domain to be non-conformally represented by the mesh. The boundary cuts the background mesh arbitrarily, and the latter is locally refined, usually using oct-tree based techniques – which also simplify parallelisation, in particular using modern graphical processing units (GPUs). The method was recently revisited within the context of the extended finite element method [22, 25, 99].

- **Use of methods only requiring a boundary discretisation:** These methods use semi-analytical ways to decrease the dimension of the problem. The examples are boundary element method initiated by Rizzo [119] and the scaled boundary finite element method by Song and Wolf [137]. Boundary element methods rely on fundamental solutions, which are not available for the non-linear problems and heterogeneous materials. On the other hand, scaled boundary finite element method can be extended to non-linear materials but not suitable for complex geometries where the scaling centre cannot be easily specified.

This paper, is focused on notions related to isogeometric analysis, in particular to the isogeometric finite element method (IGAFEM), the isogeometric boundary element method (IGABEM), the NURBS-enhanced finite element method (NEFEM) and some of the competing methods, in particular immersed or implicit boundary techniques.

The paper is organised as follows: first, the technology that underpins CAGD and which forms the basis of IGA is outlined; next, the basic concepts of IGA and the fundamental differences to conventional implementations are described; recent developments in IGA are then described, emphasising the beneficial properties over traditional approaches and finally, a comparative review with other methods which can also alleviate the mesh generation is given, followed by the comments on future directions and problems that IGA faces.

2 Computer aided geometric design technology

Isogeometric analysis (IGA) [70] has been successful at bringing two disparate research communities together: that of computer aided geometric design (CAGD) and numerical analysis. As argued above, converting CAD models into a form suitable for analysis is considered a laborious task, often dominating the design process. IGA offers a new direction, where, instead, a much more direct route is taken allowing analysis to proceed much more simply than before. This is achieved by using the data provided by CAD models *directly* rather than converting it through a preprocessing routine into a form suitable for analysis. IGA has found that the functions which are used to describe CAD surfaces are often directly amenable to analysis where, for example, properties such as partition of unity (PUM), linear independence, non-negativity and weak Kronecker delta property can be proven.

To the authors' knowledge, the first work of integrating CAD and engineering analysis can be traced back to the paper by Kagan *et al.* [75], where B-splines were used as basis functions to represent both the geometry and the unknown fields. Following this idea, Cirak *et al.* [38] proposed a paradigm for thin-shell analysis, but used subdivision surfaces instead of B-splines. These ideas were formalised and generalised by the acclaimed work of Hughes *et al.* [70] based on NURBS, which has been extended to T-splines [12] recently. The recent work by Nguyen-Thanh [106, 105] and Wang [149] has incorporated PHT-splines.

Goals	B-splines	NURBS	Analysis-suitable T-splines	PHT-splines	Subdivision surfaces
Topology flexibility	No	No	Yes	Yes	Yes
Watertight geometry	No	No	Yes	Yes	Yes
Smoothness/Continuity	desired continuity	desired continuity	desired continuity	C^2	C^2 or C^1
Partition of unity	Yes	Yes	Yes	Yes	N/A
Kronecker delta property	weak	weak	weak	weak	N/A
Local refinement	No	No	Yes	Yes	Yes
Linear independence	Yes	Yes	Yes	Yes	N/A

Table 2: Comparison of various CAD descriptions from a geometrical and analysis point of view

The current section is devoted to providing a non-exhaustive overview of the basic technology provided by the computer aided geometric design community, as this technology is central to isogeometric analysis. This overview is performed whilst keeping in mind that the information provided during CAD is to be used for analysis purposes. Table 2 gives a comparison of the technology reviewed here, namely B-splines, non-uniform rational B-splines (NURBS), analysis-suitable T-splines, PHT-splines and subdivision surfaces. Other very recent techniques which are not reviewed here include RHT-splines.

2.1 B-splines

B-spline geometry is a mapping from parametric space to physical space through a linear combination of B-spline basis functions, which are defined in parametric space, and the corresponding coefficients which are called control points because their physical meaning are a series of points scattered in physical space. B-splines in one dimension can be expressed as:

$$\mathbf{C}(\xi) = \sum_{I}^{n} N_{I,p}(\xi)\mathbf{B}_I \tag{1}$$

where $\mathbf{C}(\xi)$ denotes the physical curve of interest, ξ the coordinate in parametric space, B_I the control points, $N_{I,p}$ the B-spline basis functions of order p.
B-splines possess the following properties:

- **The convex hull property:** The B-splines geometry is contained in the convex hull constructed by the control grid, which is a mesh interpolated by control points.

- **The variation diminishing property:** No plane has more intersections with the curve than it has with the control grids. This property renders B-splines less oscillatory than Lagrangian polynomials.
- **The transformation invariance property:** An affine transformation to a B-splines curve can be achieved by applying an affine transformation to the control points.
- **The partition of unity property:**

$$\sum_{I=1}^{n} N_I(\xi) = 1 \tag{2}$$

- **The non-negative property:** Each basis function is pointwise non-negative over the domain.
- **Ease of refinement and control of continuity:** It is possible to use the intrinsic building blocks of NURBS for local refinement in the standard h and p way, but also by elevating the order of basis functions and continuity, simultaneously, also known as k refinement (see Section 3.2).
- **Locally supported:** The basis functions are pointwise non-negative in $p + 1$ knot spans. This property leads to a fact that the modification of a control point only influences the geometry locally.
- **Weak Kronecker delta property:** A weak Kronecker delta property means $N_I(\mathbf{x}) = 0$ but $N_I(\mathbf{x}_I) \neq 1$, which is useful for enforcing boundary conditions in engineering analysis, because only the control points corresponding to boundaries need to be considered.

2.2 Non-uniform rational B-splines

Non-uniform rational B-splines (NURBS) [115, 124] are an extension of B-splines introducing the weights for basis functions to represent a wider range of geometries such as conic sections. From a geometric point of view, NURBS in $\mathbb{R}^d$ is a projective transformation of B-splines in $\mathbb{R}^{d+1}$. NURBS inherit the properties of B-splines as mentioned above, but still have some drawbacks:

- **Rational functions:** As NURBS are not polynomial functions, integrating them cannot be done exactly using Gauß quadrature.
- **Tensor product:** The parametric space and control points rely on a structured grid due to the tensor product property of NURBS, which increases the redundancy of the degrees of freedom.
- **Continuity:** NURBS usually achieve only C^0 continuity between the patches.

- **Geometry repair:** From a computational geometry point of view, NURBS based geometry always requires some level of repair due to gaps or overlaps of the various patches making up a geometry.

The following subsections summarise a few of the recent advances to alleviate some of those shortcomings.

2.3 Subdivision of surfaces

Subdivision surfaces, firstly proposed by Catmull and Clark [35], can possess a topology flexibility and ability to construct a watertight geometry. Subdivision surfaces use a recursive rule to interpolate or approximate a smooth surface. The methods can be divided into two classes:

- **Interpolating schemes:** When meshes are refined, new vertices are added without changing the vertices in the coarser mesh. Consequently, all of the vertices produced during subdivision will interpolate the surface. Interpolating schemes for quadrilateral meshes have been introduced by Kobbelt *et al.* [79], while Dyn [58] and Zorin *et al.* [163] described interpolating schemes for triangular meshes. In both cases the limit surfaces are C^1 but their curvatures do not exist.

- **Approximation schemes:** In approximation schemes, in contrast to interpolating schemes, the vertices in the coarse mesh need to be recomputed in refined mesh by adding the new vertices. The examples are the schemes proposed by Catmull and Clark [35] and Doo and Sabin [56], which are the first subdivision surface schemes and use quadrilateral meshes. Loop [90] proposed a scheme based on triangular meshes. Theses approximation schemes can achieve C^2 continuous, except in some isolated points which are C^1. After parametrisation, subdivision surfaces can be incorporated into IGA [38, 37, 39].

2.4 T-splines

T-splines were proposed by Sederberg *et al.* [126] to overcome the drawbacks of NURBS. T-spline control grids, also called T-meshes, permit T-junctions which are similar to the concept of the "hanging nodes" and oct or quad-tree meshes in the FEM. In this way, lines of control points need not traverse the entire control grid. If a T-mesh is simply a rectangular grid with no T-junctions, the T-spline reduces to a B-spline. T-splines support many valuable operations within a consistent framework, such as local refinement, and the merging of several B-spline surfaces that have different knot vectors into a single gap-free model. T-splines form a subset of PB-splines, *i.e.* the basis functions are defined on local knot vectors. However, the local knot vector of T-splines can be inferred from a T-mesh, instead of being defined arbitrarily. *A posteriori* error estimation techniques for local h-refinement with T-splines have

been introduced very recently by Dorfel *et al.* [57]. Bezilevs *et al.* [12] have tested T-splines on some elementary two-dimensional and three-dimensional fluid and structural analysis problems.
The major drawback of T-splines is a significant implementation complexity. In addition, T-splines lose the property of linear independences, which is useful in engineering analysis. But an analysis-suitable T-splines [82], a subset of T-splines, can always satisfy this requirement.

2.5 PHT-splines

PHT-splines (polynomial splines over hierarchical T-meshes) proposed by Deng *et al.* [55] are constructed on the basis of T-splines and thus inherit their beneficial properties. PHT-splines use a hierarchical T-mesh which has a nested structure and different levels. PHT-splines define a cubic B-spline basis function in a T-mesh, which is level 0, and then modify the basis functions level by level. PHT-splines possess several main merits compared to T-splines:

1. The basis functions of PHT-splines are polynomial.
2. The local refinement is simpler compared to T-splines.
3. The conversion between NURBS and PHT-splines is very fast.

However, PHT-splines are only C^1 continuous, which are not sufficient to represent complex geometries. PHT-splines contain a closed set (linearly independent) of basis functions, which are important for combining them with FEM.

3 Isogeometric analysis

CAD models provide a complete description of the geometry of a problem, but for analysis to be carried out there is usually a need to carry out some level of preprocessing to arrive at a model suitable for analysis. This is also the case in isogeometric analysis (IGA), although the chief motive of this method is to simplify this preprocessing step.

The difference between traditional practice and IGA however, is that the time and difficulty of this preprocessing step is significantly reduced. For example, in the case of IGA coupled with immersed and boundary element methods, the task of meshing is almost completely circumvented. This section aims to give an overview of the preprocessing steps required for IGA for various numerical methods and the differences and advantages over traditional techniques.

By utilising the exact geometry and noting the advantageous properties of the basis functions that are used in CAD (see Section 2.2), several important developments have been made in the computational mechanics community. Moreover, since the modifications required to convert a traditional numerical analysis code into one that utilises the same basis functions as CAD are often minimal, the method can be adopted quickly and applied to existing numerical methods. Several of these recent developments are highlighted in the present section where the benefits and drawbacks compared to traditional approaches are demonstrated.

3.1 Motivation

As discussed briefly, the main purpose of isogeometric analysis (IGA), presented by Hughes *et al.* [70] is to increase the ties between computer aided design (CAD) data and the numerical method of choice for analysis. In the original paper, this method was the Galerkin finite element method (FEM).

For simplicity, the example of linear elastic structural mechanics problems will be considered. In engineering practice, the geometry is described by CAD packages allowing the construction of very complex three-dimensional geometries, usually described by several NURBS (non-uniform rational B-splines) patches. This geometry then needs to be meshed[1] and analysed (*e.g.* by the FEM) to obtain an approximate distribution of the unknown fields within the structure. This analysis is used by the engineer to assess the design. If alterations to this design are required, they must then be communicated to the drafting team, and the geometry of the component suitably modified. A new mesh must then be regenerated by the analysis team to perform a second stress analysis, and the process is repeated as many times as is required for a suitable design to be obtained.

Considering this design cycle, it becomes clear that tying the geometrical CAD data to the mesh in an automatic and direct way, where the mesh would be defined automatically from the CAD data is highly desirable. This desirable integration is evident from the consolidation trends observed in the CAD and Analysis industry where CAD companies join forces with leaders in analysis, with the goal to streamline the design cycle and decrease lead time in computer aided design.

In IGA the geometry can be represented exactly for analysis by direct use of CAD data through the use of NURBS-based approximations. This approach is being pursued widely both in the engineering community by improving the basis functions and seeking adaptive methods [83, 12, 57] and applying the ideas to fields outwith mechanics, such as electromagnetics [33]. The idea is also being taken up by the applied mathematics community through the derivation of error estimates by, *e.g.* [21]. This engineering and mathematics research has allowed a rather rapid development of the method through application-oriented investigations and sound mathematical analysis, respectively.

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[1]This meshing operation is inexact, and only as the mesh size goes to zero is the geometry exactly represented. This geometry approximation error however converges to zero faster than the approximation error on the unknown fields.

3.2 Methods of refinement

One refinement method is known as knot insertion, which is analogous to h-refinement in FEM and relies on splitting an element. However, the insertion of the existing values can reduce the continuity of basis function without producing new elements. The second is order elevation, which has similarities with p-refinement in classical FEM. Finally, k-refinement has no analogue in FEM, because it allows increasing the order of the basis functions and the number of elements, simultaneously. The advantage of this method is that fewer degrees of freedom are introduced compared to p-refinement. [42] details the h-p-k- refinement schemes, Dorfel *et al.* [57] explains an adaptive h-refinement, and Beirão da Veiga *et al.* [20] introduce error estimates for h-p-k- refinement schemes.

3.3 *A priori* error estimation

The difficulties of *a priori* error estimation in isogeometric analysis arises from two aspects:

1. NURBS are not interpolatory.
2. The support of NURBS basis functions are in general found to be larger than polynomial approximations of an equivalent order.

The first difficulty is solved by pulling back the approximated function to the regular parametric space and noting that NURBS in $\mathbb{R}^d$ is a projective transformation of B-splines in $\mathbb{R}^{d+1}$. The second difficulty is addressed by introducing the so-called "bent" Sobolev spaces [16]. A meaningful approximation expression is derived to include not only the approximated function but also the gradient of the mapping, which does not change with the refinement.

Finally, a significant result is that: the IGA result using p-order NURBS has the same order of convergence as the Finite element method using p-order polynomials.

3.4 Bézier extraction

To further integrate isogeometric analysis with existing FEM codes, Bézier extraction is utilised for NURBS [31] and T-splines [123] to allow numerical integration of smooth functions to be performed on C^0 Bézier elements. The idea of Bézier extraction is that localised NURBS or T-spline basis functions can be represented by a linear combination of the Bernstein polynomials. In addition to localising the support of basis functions into an element, Bézier extraction provides an element data structure suitable for analysis. That is, similar to Lagrangian polynomials in traditional FEM, Bernstein polynomials do no change element by element. Furthermore, no intermediate parametric space is employed, hence the physical geometry can be mapped directly from parent elements. However, it should be noted that Bézier extraction increases the degrees of freedom of the system.