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Abstract

We study the behavior of maximal geodesics outside a sufficiently large compact set in a finitely connected, complete and noncompact 2-dimensional Riemannian manifold (possibly with boundary). The total curvature of such a manifold was first investigated by S. Cohn-Vossen [Co1, Co2]. Assume for simplicity that a complete manifold M is homeomorphic to $\mathbf{R}^2$. He proved in [Co1] that if the total curvature of M exists, it is less or equal to 2π. One of our main results (see Theorems B and C in 2.3) states that if the total curvature of M exists and is strictly less than 2π, then any maximal geodesic outside a sufficiently large compact set in M forms almost the shape as that of a maximal geodesic in a flat cone, and its rotation number (originally due to H. Whitney [Wh]) is controlled by the total curvature.

Key Words and Phrases: geodesics, the Gauss-Bonnet formula, total curvature.

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Introduction

Recall that a topological space is said to be finitely connected if its i-th homotopy groups are finitely generated for all i (cf. [Bu, §29.6]). Let M be a finitely connected 2-dimensional smooth manifold with finitely connected (but not necessarily compact) piecewise smooth boundary ∂M. Then, M has finitely many ends e_j, where $1 \leq j \leq k$, (i.e. M has k boundary components in the sense of [AS, Chap. I, 36C]). If we compactify M by adding at infinity points in 1-1 correspondence with the e_j's, we get a compact topological manifold with boundary $M' := M \cup \{e_1, \dots, e_k\}$. The ends e_j's are of two types according to whether e_j belongs to the boundary $\partial M'$ or to the interior $\mathrm{Int}(M')$ of M'. In addition suppose that M is a complete Riemannian manifold. A complete (or as we prefer to say a maximal) geodesic of M is said to be *distant* if it is entirely contained into a sufficiently small neighborhood of some e_j. Assume that both the total curvature $c(M)$ of M and the total geodesic curvature $\kappa(M)$ of ∂M with respect to M are defined and that the sum $\kappa(M) + c(M)$ has a meaning. Then the *curvature at infinity* $\kappa_\infty(M)$ of M is defined to be $\kappa_\infty(M) := 2\pi\chi(M) - \kappa(M) - c(M)$, where $\chi(M)$ is the Euler characteristic of M. By a suitable generalization of Cohn-Vossen's theorem (see 3.4 below), one has $\kappa_\infty(M) \geq \pi\chi(\partial M)$, where $\chi(\partial M)$ is the number of noncompact connected components of ∂M. For each end e_j there exists an arbitrarily small neighborhood M_j' of e_j such that if $e_j \in \partial M'$, $M_j := M_j' - \{e_j\} \subset M$ is a Riemannian half plane; if $e_j \in \mathrm{Int}(M')$, $M_j = M_j' - \{e_j\}$ is a Riemannian half cylinder (see 3.4.2 and 3.4.3 for definitions of Riemannian half planes and half cylinders). By the Gauss-Bonnet theorem, each number $\kappa_\infty(M_j)$ is depending only on e_j but independent of M_j, and is called the *curvature at e_j*. Applying Cohn-Vossen's theorem to each M_j, one has

$$\kappa_\infty(M_j) \geq \begin{cases} \pi & \text{if } e_j \in \partial M' \\ 0 & \text{if } e_j \in \mathrm{Int}(M') \end{cases}$$

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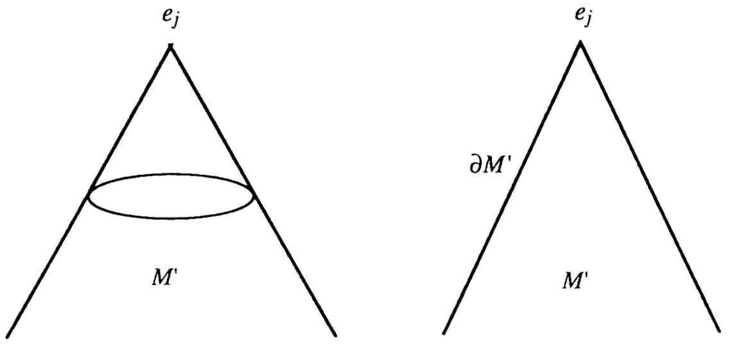
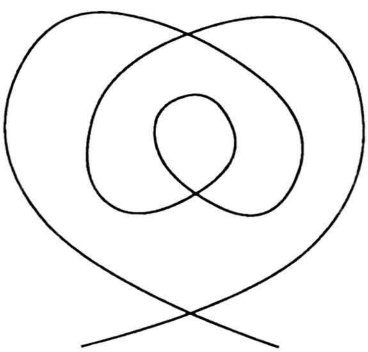
and moreover, by Gauss-Bonnet theorem,

$$\kappa_\infty(M) = \sum_{j=1}^{k} \kappa_\infty(M_j).$$

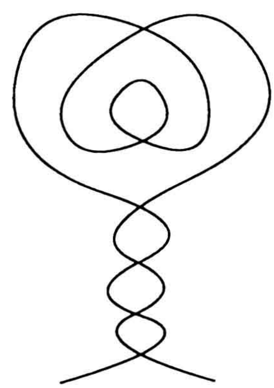


When $e_j \in \partial M'$, all maximal geodesics close enough to e_j are simple; if $\kappa_\infty(M_j) < 2\pi$, then close enough to e_j there exist no maximal geodesics; if $\kappa_\infty(M_j) = 2\pi$, one can not say whether such geodesics exist or not; if $\kappa_\infty(M_j) > 2\pi$, there exists maximal geodesics arbitrarily close to e_j (see Conclusion 7.3). When $e_j \in \mathrm{Int}(M')$, if $\kappa_\infty(M_j) = 0$, nothing can be said in the absence of further assumptions (see Remarks and Examples in 2.2 and 2.3); if $\kappa_\infty(M_j) > 0$, there exist maximal geodesics arbitrarily close to e_j (see Corollary to Theorem A in 2.2 and Conclusion 7.3); if $\kappa_\infty(M_j) \in (0,+\infty)$, then close enough to e_j maximal geodesics essentially behaved as those of a cone having vertex angle equal to $\kappa_\infty(M_j)$ in a sense made precise in the Preliminary Remark of 2.3 (see also Conclusion 7.3); if $\kappa_\infty(M_j) = +\infty$, then close enough to e_j all maximal geodesics are simple (see Theorem B in 2.3 and Conclusion 7.3). Since each M_j is either a Riemannian half plane or a Riemannian half cylinder and since, according to Remark 3.4.3.2, any Riemannian half cylinder can be isometrically embedded into a Riemannian plane, the previous general statements are mere consequences of the main results of this paper concerning Riemannian planes stated in Chapter 2 and proved in Chapters 4, 5 and 6 and of additional results concerning Riemannian half planes (see 3.8 and 7.2, see also 7.1 for half cylinders).

In the case where M is a Riemannian plane (i.e. homeomorphic to $\mathbf{R}^2$) with the previously introduced notations $\kappa_\infty(M) = 2\pi - c(M) \in [0,+\infty]$, Theorems B and C stated in Chapters 2 and 3 assert that, if $\kappa_\infty(M) > 0$, outside a sufficiently large contractible compact subset K of M (called 'fat enough' in Chapter 5), all maximal geodesics in M are regular in a sense given in Chapter 1, which means that they behave as those of a cone, of a two-sheeted hyperboloid or of a paraboloid. In general, the number of double points of such geodesics (which have no triple points) is equal to $\mathrm{intg}(\pi / \kappa_\infty(M))$ (where $\mathrm{intg}(\cdot)$ means the integral part). However, when $\pi / \kappa_\infty(M)$ is an integer, it may happen that this number drops down to $\pi / \kappa_\infty(M) - 1$ ('the less crossing situation'). Moreover in 'the more crossing situation', i.e. when the number of double points is equal to $\pi / \kappa_\infty(M)$, it may happen that the geodesic crosses itself over and over again in such a way that it can no longer be called 'regular' but only 'almost regular' (see 1.10).



a reguler geodesic



an almost regular geodesic

It is interesting in itself (and also useful for the proof) to notice that, if we partly relax the constrains on the quantities both of positive and of negative curvature that a contractible compact subset must contain in order to be called fat enough, one obtains a larger class of subsets K of M called 'fat' outside which all maximal geodesics are semi-regular in a sense given in 1.4 (see Chapter 4). Theorem D (see 2.3) and its corollaries give additional results in the case where M does not have too much positive curvature and in the case where M has no negative curvature. Corollary to Theorem A asserts that a Riemannian plane has maximal geodesics arbitrarily close to infinity, a result which guarantees that Theorems B, C and D are not empty. Theorem A stated in 2.2 and proved in Chapter 6 asserts that the visual diameter of any compact set $K \subset M$ looked at from a point $p \in M$ tends to zero when p tends to infinity. It seems that it is not possible to prove Corollary to Theorem A without a control of visual diameters. Moreover such a result is extended to that for unbounded K, which implies that the number $\kappa_\infty(H)$ controls whether in a Riemannian half plane H a maximal geodesic arbitrarily close to infinity exists or not (see 7.2). The result about the visual images of unbounded K will be published independently.

Since all the proofs given in the present paper are derived from the Gauss-Bonnet formula and from Cohn-Vossen's theorem, the statements of this paper clearly extend to G-surfaces in the sense of Busemann (see [Bu]). Although new, our results should be considered as elementary. For this reason our presentation tried to be as self-contained as reasonably possible in order to make the article accessible to beginners.

In Chapter 1 we defined semi-regular, almost regular and regular curves in order to be able in Chapter 2 to state the main results concerning Riemannian planes. In Chapter 3 we introduce a suitable notion of boundary in order to generalize the Gauss-Bonnet theorem. In the same chapter we also generalize Cohn-Vossen's theorem to a large class of complete 2-dimensional Riemannian manifolds. Chapter 4 (resp. 5) shows that outside a fat (resp. fat enough) subset of a Riemannian plane M, all maximal geodesics are semi-regular (resp. almost regular with suitable index). Chapter 6 proves Theorem A (the statement concerning visual diameter). Chapter 7 generalizes the previous results to finitely connected Riemannian manifolds with finitely connected boundary.

1. The semi-regular curves in a differentiable plane

In this paragraph we introduce preliminary notions needed in order to state the main results of Chapter 2.

1.1. Proper transversal immersion. Let M be any smooth surface. A differentiable mapping α of a (not necessarily compact) interval I of $\mathbf{R}$ into M is said to be a *weakly transversal immersion* (resp. a *transversal immersion*) if it satisfies Conditions (i) and (ii) (resp. Conditions (i), (ii) and (iii)).

(i) (immersibility condition) $\dot{\alpha}(t) := \frac{d}{dt}\alpha(t) \neq 0$ for all $t \in I$.

(ii) (source transversality condition) whenever $\alpha(a) = \alpha(b) = p$ for $a \neq b$, the tangent vectors $\dot{\alpha}(a)$ and $\dot{\alpha}(b)$ are linearly independent in T_pM.

(iii) (target transversality requirement) The mapping α has no triple points, in other words there exist no $a, b, c \in I$ such that $a \neq b \neq c \neq a$ and that $\alpha(a) = \alpha(b) = \alpha(c)$.

1.1.1. Lemma. Let α be a proper transversal immersion of a not necessarily compact interval I into a smooth surface M. Then the set of double points of α is a discrete subset of M.

□

Suppose from now on that the surface M is diffeomorphic to $\mathbf{R}^2$. Assume for convenience that M is oriented and suppose that $a < b$. A crossing point $p = \alpha(a) = \alpha(b)$ of α will be said to have a *positive sign* ($\text{sgn}(p) = 1$) when the basis $(\dot{\alpha}(a), \dot{\alpha}(b))$ has positive orientation and a *negative sign* ($\text{sgn}(p) = -1$) otherwise.

1.2. Definition. Suppose $I = \mathbf{R}$ and let $n_+(\alpha)$ (resp. $n_-(\alpha)$) be the number (possibly infinite) of double points having positive (resp. negative) sign of a proper transversal immersion α: $\mathbf{R} \to M$ and let

$$n(\alpha) = \limsup_{\substack{s \to -\infty \\ t \to +\infty}} | n_+(s,t) - n_-(s,t) |,$$

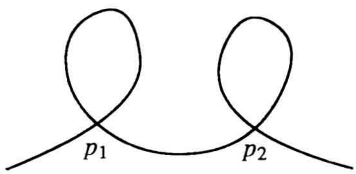
where $n_+(s,t)$ (resp. $n_-(s,t)$) denotes the number of *positive* (resp. *negative*) double points of the closed arc $\alpha|[s,t]$. Whenever α is such that these three quantities are not all equal to infinity, the *rotation number* $\mathrm{rot}(\alpha) \in \mathbf{N} \cup \{\infty\}$ is defined to be the presently introduced quantity $n(\alpha)$, (where $\mathbf{N}$ denotes the set of nonnegative integers).

Remark. Notice that $n(\alpha)$ (and therefore $\mathrm{rot}(\alpha)$ whenever it is defined) does not depend on the chosen orientation of M, so that the notion of rotation number makes sense even when M is not assumed to be oriented. Notice also that $n(\alpha)$ does not depend on the parameterization of α and is an invariant of the compactly supported regular homotopy class of α. Recall that a proper transversal differentiable immersion α: $\mathbf{R} \to M$ is said to be *compactly supported regular homotopic* to the proper transversal immersion β when $\alpha(t) = \beta(t)$ for all t outside some open interval $(a,b) \subset \mathbf{R}$ and when there exists a regular homotopy between $\alpha|[a,b]$ and $\beta|[a,b]$ fixing $\dot{\alpha}(a)$ and $\dot{\alpha}(b)$.

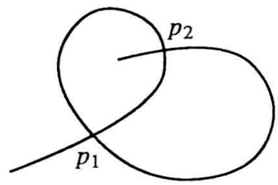
1.3. The order relation between double points. Let α be a proper transversal immersion of some (not necessarily compact) closed interval of $\mathbf{R}$ into M and let p_1 and p_2 be two double points of α such that $p_i = \alpha(a_i) = \alpha(b_i)$ with $a_i < b_i$ for $i = 1,2$. Set $p_1 \le p_2$ whenever $[a_1,b_1] \subset [a_2,b_2]$. This convention defines a partial order relation on the set of double points of α such that for each double point p there exists at least one minimal double point q such that $q \le p$. Notice that, when both the relations $p_1 \le p_2$ and $p_2 \le p_1$ do not hold, one of the two following situations may occur.

(i) $[a_1,b_1] \cap [a_2,b_2] = \emptyset$, in which case p_1 and p_2 are said to be *independent*.

(ii) $[a_1,b_1] \cap [a_2,b_2]$ is a nonempty interval $I \neq [a_i,b_i]$ for $i = 1,2$, in which case p_1 and p_2 are said to *link*.

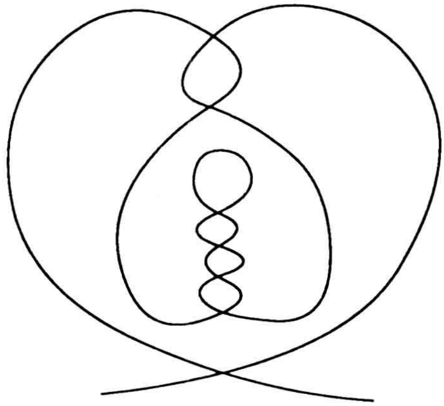


p_1 and p_2 are independent.



p_1 and p_2 link.

1.4. Definition. A proper transversal immersion of some (not necessarily compact) closed interval of $\mathbf{R}$ into M is called a *semi-regular arc* when the set of double points of α is totally ordered by the previously defined order relation.



A semi-regular arc

In other words, setting $\overline{n}(\alpha) := n_+(\alpha) + n_-(\alpha)$ and for all $n \in \mathbf{N} \cup \{\infty\}$ denoting by $[n]$ the set of all integers i such that $1 \leq i \leq n$, the proper transversal immersion α is said to be semi-regular when there exists a sequence of closed