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Stochastic Control and Mathematical Modeling

Applications in Economics

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This is a concise and elementary introduction to stochastic control and mathematical modeling. This book is designed for researchers in stochastic control theory studying its application in mathematical economics and those in economics who are interested in mathematical theory in control. It is also a good guide for graduate students studying applied mathematics, mathematical economics, and nonlinear PDE theory.

Contents include the basics of analysis and probability, the theory of stochastic differential equations, variational problems, problems in optimal consumption and in optimal stopping, optimal pollution control, and solving the Hamilton–Jacobi–Bellman equations with boundary conditions. Major mathematical requisitions are contained in the preliminary chapters or in the appendix so that readers can proceed without referring to other materials.

Hiroaki Morimoto is a Professor in Mathematics at the Graduate School of Science and Engineering at Ehime University. His research interests include stochastic control, mathematical economics and finance and insurance applications, and the viscosity solution theory.

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To My Teacher M. Nisio

Preface

The purpose of this book is to provide a fundamental description of stochastic control theory and its applications to dynamic optimization in economics. Its content is suitable particularly for graduate students and scientists in applied mathematics, economics, and engineering fields.

A stochastic control problem poses the question: what is the optimal magnitude of a choice variable at each time in a dynamical system under uncertainty? In stochastic control theory, the state variables and control variables, respectively, describe the random phenomena of dynamics and inputs. The state variable in the problem evolves according to stochastic differential equations (SDE) with control variables. By steering of such control variables, we aim to optimize some performance criteria as expressed by the objective functional. Stochastic control can be viewed as a problem of decision making in maximization or minimization. This subject has created a great deal of mathematics as well as a large variety of applications in economics, mathematical finance, and engineering.

This book provides the basic elements of stochastic differential equations and stochastic control theory in a simple and self-contained way. In particular, a key to the stochastic control problem is the dynamic programming principle (DPP), which leads to the notion of viscosity solutions of Hamilton–Jacobi–Bellman (HJB) equations. The study of viscosity solutions, originated by M. Crandall and P. L. Lions in the 1980s, provides a useful tool for dealing with the lack of smoothness of the value functions in stochastic control. The main idea used to solve this maximization problem is summarized as follows:

(a) We formulate the problem and define the supremum of the objective functional over the class of all control variables, which is called the *value function*.
(b) We verify that the DPP holds for the value function.
(c) By the DPP, the value function can be viewed as a unique viscosity solution of the HJB equation associated with this problem.
(d) The uniform ellipticity and the uniqueness of viscosity solutions show the existence of a unique classical solution to the boundary value problem of

the HJB equation. This gives the smoothness of the viscosity solution of the HJB equation.

(e) We seek a candidate of optimal control by using the HJB equation. By using Itô's formula, we show the optimality.

This book is divided into three parts: Part I – Stochastic Calculus and Optimal Control Theory; Part II – Applications to Mathematical Models in Economics; and Part III – a collection of appendices providing background materials.

Part I consists of Chapters 1–5. In Chapter 1, we present the elements of stochastic calculus and SDEs, and in Chapter 2, we present the formulation of the weak solutions of SDEs, the concept of regular conditional probability, the Yamada–Watanabe theorem on weak and strong solutions, and the Markov property of a solution of SDE.

In Chapter 3, we introduce the DPP to issue (b). The verification of the DPP is rather difficult compared to the deterministic case. The Yamada–Watanabe theorem in Chapter 2 makes its proof exact. The supremum of (a) is taken over all systems in the weak sense.

Chapter 4 provides the theory of viscosity solutions of the HJB equations for (c). Using Ishii's lemma, we show the uniqueness results on viscosity solutions.

Chapter 5 is devoted to the boundary value problem of the HJB equations for (d) in the classical sense. Section 5.4 explains how to apply (a)–(e) in stochastic control.

Part II consists of Chapters 6–12. Here we present diverse applications of stochastic control theory to the mathematical models in economics. In Chapters 6–10, we take the state variables in these models as the remaining stock of a resource, the labor supply, and the price of the stock. The criteria in the maximization procedure are often given by the utility function of consumption rates as the control variables. Along (a)–(e), an optimal control is shown to exist.

Chapters 11 and 12 deal with the linear and nonlinear variational inequalities, instead of the HJB equations, which are associated with the stopping time problem. The variational inequality is analyzed by the viscosity solutions approach for optimality.

Part III consists of Appendices A–H. These provide some background material for understanding stochastic control theory as quickly as possible.

The prerequisites for this book are basic probability theory and functional analysis (see e.g., R. B. Ash [2], H. L. Royden [139], and A. Friedman [69]). See M. I. Kamien and N. L. Schwartz [80], A. C. Chiang [33], A. K. Dixit and R. S. Pindyck [46], L. Ljungqvist and T. J. Sargent [107], and R. S. Merton [114], for economics references.

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H. Morimoto
Matsuyama
January 2009

Contents

Part I

Stochastic Calculus and Optimal Control Theory

1

Foundations of Stochastic Calculus

We are concerned here with a stochastic differential equation,

$$dX(t) = b(X(t))dt + \sigma(X(t))dB(t), \quad t \geq 0,$$
$$X(0) = x \in \mathbf{R}^N,$$

in N-dimensional Euclidean space $\mathbf{R}^N$. Here b, σ are Lipschitz functions, called the *drift term* and the *diffusion term*, respectively, and $\{B(t)\}$ is a standard Brownian motion equation defined on a probability space $(\Omega, \mathcal{F}, P)$. This equation describes the evolution of a finite-dimensional dynamical system perturbed by noise, which is formally given by $dB(t)/dt$. In economic applications, the stochastic process $\{X(t)\}$ is interpreted as the labor supply, the price of stocks, or the price of capital at time $t \geq 0$. We present a reasonable definition of the second term with uncertainty and basic elements of calculus on the stochastic differential equation, called *stochastic calculus*.

A. Bensoussan [16], I. Karatzas and S. E. Shreve [87], N. Ikeda and S. Watanabe [75], I. Gihman and A. Skorohod [72], A. Friedman [68], B. Øksendal [132], D. Revuz and M. Yor [134], R. S. Liptzser and A. N. Shiryayev [106] are basic references for this chapter.

1.1 Review of Probability

1.1.1 Random Variables

Definition 1.1.1. *A triple* $(\Omega, \mathcal{F}, P)$ *is a* probability space *if the following assertions hold:*

(a) Ω *is a set.*
(b) $\mathcal{F}$ *is a* σ*-algebra, that is,* $\mathcal{F}$ *is a collection of subsets of* Ω *such that*
  (i) $\Omega,\ \phi \in \mathcal{F}$,
  (ii) *if* $A \in \mathcal{F}$, *then* $A^c := \Omega \setminus A \in \mathcal{F}$,
  (iii) *if* $A_n \in \mathcal{F}, n = 1, 2, \ldots,$ *then* $\bigcup_{n=1}^{\infty} A_n \in \mathcal{F}$.

(c) *P is a probability measure, that is, a map $P : \mathcal{F} \to [0, 1]$, such that*
  (i) $P(\Omega) = 1$,
  (ii) *if $A_n \in \mathcal{F}, n = 1, 2, \ldots,$ disjoint, then* $P(\bigcup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} P(A_n)$.

Definition 1.1.2. *A probability space $(\Omega, \mathcal{F}, P)$ is complete if $A \in \mathcal{F}$ has $P(A) = 0$ and $B \subset A$, then $B \in \mathcal{F}$ (and, of course, $P(B) = 0$), that is, $\mathcal{F}$ contains all P-null sets.*

Remark 1.1.3. *Any probability space $(\Omega, \mathcal{F}, P)$ can be made complete by the completion of measures due to Carathéodory. We also refer to the proof of the Daniell Theorem, Theorem 2.1 in Chapter 2.*

Definition 1.1.4. *For any collection $\mathcal{G}$ of subsets of Ω, we define a smallest σ-algebra $\sigma(\mathcal{G})$ containing $\mathcal{G}$ by*

$$\sigma(\mathcal{G}) = \cap\{\mathcal{F} : \mathcal{G} \subset \mathcal{F},\ \mathcal{F}\ \sigma\text{-algebra of } \Omega\},$$

which is the σ-algebra generated by $\mathcal{G}$.

Example 1.1.5. *On the set of real numbers $\mathbf{R}$, we take $\mathcal{G} = \{$open intervals$\}$ and denote by $\mathcal{B}(\mathbf{R})$ the σ-algebra $\sigma(\mathcal{G})$ generated by $\mathcal{G}$, which is the Borel σ-algebra on $\mathbf{R}$.*

Definition 1.1.6. *Let $(\Omega, \mathcal{F}, P)$ be a complete probability space.*

(a) *A map $X : \Omega \to \mathbf{R}$ is a random variable if*

$$X^{-1}(B) := \{\omega : X(\omega) \in B\} \in \mathcal{F}, \qquad \text{for any } B \in \mathcal{B}(\mathbf{R}).$$

(b) *For any random variable X, we define the σ-algebra $\sigma(X)$ generated by X as follows:*

$$\sigma(X) = \sigma(\mathcal{G}) = \mathcal{G}, \qquad \mathcal{G} := \{X^{-1}(B) : B \in \mathcal{B}(\mathbf{R})\} \subset \mathcal{F}.$$

Proposition 1.1.7. *Let X, Y be two random variables. Then Y is $\sigma(X)$ measurable if and only if there exists a Borel measurable function $g : \mathbf{R} \to \mathbf{R}$ such that*

$$Y(\omega) = g(X(\omega)), \qquad \text{for all } \omega \in \Omega.$$

Proof. Since $Y = Y^+ - Y^-$, we will show the "only if" part when $Y \geq 0$.

(1) Suppose that Y is a simple random variable. Then Y is of the form:

$$Y(\omega) = \sum_{i=1}^{n} y_i 1_{F_i}(\omega),$$

where $y_i \geq 0$, $F_i \in \sigma(X)$ and the F_i are pairwise disjoint. By definition, there exists $D_i \in \mathcal{B}(\mathbf{R})$, for each i, such that $F_i = X^{-1}(D_i)$. Clearly, the D_i are pairwise disjoint. Define

$$g(y) = \begin{cases} y_i, & y \in D_i, \\ 0, & y \notin \bigcup_{i=1}^{n} D_i. \end{cases}$$

Then

$$Y(\omega) = \sum_{i=1}^{n} y_i 1_{\{X^{-1}(D_i)\}}(\omega) = \sum_{i=1}^{n} y_i 1_{D_i}(X(\omega)) = g(X(\omega)).$$

(2) In the general case, there exists a sequence of simple random variables Y_n converging to Y. Let g_n be the corresponding sequence of measurable functions such that $Y_n = g_n(X)$. Define

$$g(y) = \liminf_{n\to\infty} g_n(y).$$

Then g is $\mathcal{B}(\mathbf{R})$ measurable and

$$Y(\omega) = \liminf_{n} Y_n(\omega) = \liminf_{n} g_n(X(\omega)) = g(X(\omega)), \quad \omega \in \Omega. \quad \blacksquare$$

1.1.2 Expectation, Conditional Expectation

Definition 1.1.8. *Let X be a random variable. The quantity*

$$E[X] = \int_{\Omega} X(\omega) dP(\omega)$$

is the expectation of X, where $E[X^+]$ or $E[X^-]$ is finite.

Definition 1.1.9. *Let X, Y be two random variables on a complete probability space $(\Omega, \mathcal{F}, P)$.*

(a) *The expression $X = Y$ will indicate that $X = Y$ a.s., that is, $P(X \neq Y) = 0$.*
(b) *For $1 \leq p < \infty$, the norm $\|X\|_p$ of X is defined by*

$$\|X\|_p = (E[|X|^p])^{1/p}.$$

(c) *If $p = \infty$, then*

$$\|X\|_\infty = ess\ sup|X| = \inf\{\sup_{\omega\notin N} |X(\omega)| : N \in \mathcal{F}, P(N) = 0\}.$$

(d) *The L^p spaces are defined by*

$$L^p = L^p(\Omega) = \{X : random\ variable, \|X\|_p < \infty\}.$$

Proposition 1.1.10.

(i) *$L^p(\Omega)$ is a Banach space, that is, a complete normed linear space, for $1 \leq p \leq \infty$.*
(ii) *$L^2(\Omega)$ is a Hilbert space, that is, a complete inner product space, with inner product $(X, Y) = E[X \cdot Y]$, $\quad X, Y \in L^2(\Omega)$.*

For the proof, see A. Friedman [69, chapter 3].

Definition 1.1.11. *Let X_n, $n = 1, 2, \ldots$, and X be random variables.*

(a) *$X_n \to X$ a.s. if $P(X_n \to X$ as $n \to \infty) = 1$.*