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Mati Kilp
Ulrich Knauer
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Monoids, Acts and Categories

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Monoids, Acts and Categories

With Applications to Wreath Products and Graphs

A Handbook for Students and Researchers

by

Mati Kilp
Ulrich Knauer
Alexander V. Mikhalev

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Authors

Mati Kilp
Dept. of Mathematics
Tartu University
2400 Tartu
Estonia

Ulrich Knauer
Mathematik
Ossietzky Universität Oldenburg
26111 Oldenburg
Germany

Alexander V. Mikhalev
Dept. of Mechanics
and Mathematics
Moscow State University
Moscow 119899, Russia

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Foreword

About this book

The material of the *first part* of this book, mainly in Chapters 1 and 2, should provide a selfcontained and relatively broad scenery for (undergraduate and graduate) students working with *monoids*, *acts*, *(partial) transformations*, *congruences*, and *non-abelian categories*. At the same time this part of the book gives a background for *semigroup theory* and *automata*, *formal languages*, and other applications of semigroups. *Wreath products* of monoids, acts and categories are presented at the end of Chapter 2 in Sections 2.6 and 2.7. They may be of interest for readers with applied background and are used as applications and examples in many other places.

The *second part* of the book starting from Chapter 3 is concentrated on results on *homological classification of monoids* including *Morita type theorems* on equivalences and dualities and homological properties of monoids. This part of the text requires some advanced knowledge of algebra and, what is more important, a wide interest to discover the relations of monoids and acts with different parts of mathematics and applications.

In both parts we show how the concepts developed for acts can be applied to other areas like graphs (which with various classes of their endomorphisms form acts) and after Sections 2.6 and 2.7 to wreath products, which themselves model several other structures like endomorphism monoids of free acts or in the generalized form of Section 2.7 endomorphism monoids of arbitrary acts and cascade decomposition of automata.

It is possible to use this book for courses on representation theory of monoids, homological algebra, category theory. As far as we know this is the first monograph on representation theory of monoids and on non-additive homological algebra interpreted as the interplay between monoids and their categories of acts.

Contents in brief

The *first chapter* provides the background for the subsequent chapters. Here we deal with basic notions of semigroups and monoids. We collect quite a few results which either will be used later or seem indispensable for an understanding of semigroup theory. We introduce the concept of semigroup and monoid act and give first results and examples. We also start a systematic study of biacts and bihomomorphisms. Finally, in this chapter we give some introduction to categories and functors under the guideline of what will be needed later directly or for explaining special features of act categories compared to other algebraic categories. For a better understanding of various concepts we sometimes include examples of categories which will not be of main interest in this book.

The *second chapter* gives an account of constructions starting from the usual categorical concepts of products and coproducts up to free and cofree objects, generators and cogenerators, and to tensor products, and identifies them in act categories (and sometimes other categories). Then we proceed to wreath products of monoids and acts in Section 2.6 and to various generalizations in Section 2.7.

In the *third chapter* we go into detail with certain classes of acts, i.e. acts which enjoy special properties, thereby taking up the concepts of free and projective, cofree and injective objects, and tensor products introduced earlier mainly under categorical aspects. The logic here is to some extent self-guiding by generalizing or weakening concepts which have already been presented. On the other hand we may always look at module theory which shows striking similarities as well as striking differences.

The main guideline, however, is to get prepared for the *fourth chapter* where we give results on homological classification of monoids. There exists a big amount of such results and we present connected results in tabulated form at several places. It should be noted that all these results are describing internal properties of monoids S by external properties, namely properties of subclasses of the category of S-acts and vice versa.

In the final *fifth chapter* we present Morita theory of monoids, which as well might be considered as the basic homological problem. This is to say, we determine to which extent a monoid is characterized by its act category or subcategories thereof. In other words, how close are two monoids if they have the same act categories up to categorical equivalence (Morita equivalence of monoids) or up to dual categorical equivalence (Morita dual monoids). To be self-contained we start the chapter with some more category theory concerned with adjoint functors and go into details for act categories, thereby using a great deal of the material presented in Chapters 1 to 3. We also charaterize act categories abstractly and study endomorphism monoids of generators.

Obviously the authors have had to be selective in their choice of the material. Thus, topics that at the time of writing were of interest to them were included as well as such which had a rather developed theory.

Of course we are aware that this way we have excluded a lot of results which are also important and pretty as, for example, radical theory of semigoups and torsion theory.

Several *open problems* are given in the text together with many *exercises*.

At many places we inserted *comments* where we make historical remarks or refer to other directions and results which are related but have not been included.

A detailed list of *references* is given at the end also including some papers connected with our subjects whose contents are only mentioned in the book or may be not even that.

For a better orientation the titles of the occuring subsections are given at the beginning of every section. We made subsections rather small, so that their titles should give a relatively complete and detailed information on the contents of the book.

Definitions, lemmas etc. are numbered in every chapter by 2.1. etc. for example; when citing them, at the first place we give the chapter to which it belongs. So 2.1 from Chapter III is cited as 3.2.1.

Suggestions how selected parts of this book could be used

In Chapter 1, Sections 1.2 and 1.3 contain some basics on semigroups and monoids, Sections 1.4 and 1.5 do the same for acts over monoids. These parts can be used for courses in general algebra and are basic for rest of this book.

Sections 1.6, 1.7, 2.1 to 2.5 and 5.1 can be used to study some basics of category theory where act categories serve as important examples. Section 5.2 is a neat application characterizing act categories abstractly.

Various parts of Sections 2.1 to 2.5 will be used in the sequel continuously. Assuming some basic knowledge of these concepts it should be possible to use these parts just for references when reading Chapters 3 to 5.

Section 2.6 on wreath products will be used only in examples and exercises with the one exception of Theorem 3.17.27. Section 2.7 contains partially new generalizations from 2.6 which are used for the description of the monoid of strong endomorphism of graphs which sometimes appears in examples and exercises.

Chapter 3 can be studied in parts. The injectivity branch and the projectivity branch are rather independent of each other. On the other hand injectivity and projectivity can also be studied without the other related weaker concepts. Either approach could be the contents of an introductory course on acts (representations of monoids) possibly combined with corresponding results on homological classification in Chapter 4 or Morita equivalence in Chapter 5.

Chapter 4 contains results on homological classification of monoids, that is results of the type: all strongly flat right S-acts are free if and only if S is a group. We present all combinations of two such properties introduced in Chapter 3 for which results or partial results are known. This chapter can be seen as one basic subject of the book and is parallel to Chapter 5. We recommend to use it for study, it contains many directions for further research, in particular around flatness and also around purity. It can also be used in parts as a basis for advanced courses – the injective branch (Sections 4.1 to 4.5) and the projective branch (Sections 4.6 to 4.13 or parts thereof), the latter possibly enriched with act-regularity (Section 4.14).

We have tried to make the various interconnections clear by giving pictures and tables at appropriate places, being aware, however, that an adequate method to become familiar with the subject and to gain a deeper understanding is to construct own tables and schemes according to individually preferable systematics.

Chapter 5 needs some parts of Chapter 2, basic knowledge of categories and functors from Chapter 1 (Sections 1.6 and 1.7) and basics on acts and monoids from Chapter 1 (anybody who is familiar with these basics could start reading Chapter 5 only occasionally going back to Chapter 3 about projectives and generators needed in Sections 5.3 and 5.4, or about injectives needed in Section 5.5). So, for example Section 5.3 (with or without Section 5.5) could also be the core of a course for advanced students. Section 5.1 contains facts on adjoint functors which are needed for the rest of Chapter 5. However, they are developed broader than needed later. Here it is also possible to look up results from Section 5.1 whenever they are cited in the other parts of Chapter 5. Section 5.4, although related to various parts of the text, has a somewhat special position, it presents analogues of the Gabriel-Popescu Theorem on full embeddings, now for act categories, and on endomorphism monoids of generators in these categories.

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Introduction

The concept of a monoid, that is a semigroup with identity is relatively young. The first, often fragmentary, studies were carried out in the early twentieth century, mainly in connection with the investigation of invertible transformations which played an important role in the development of group theory. As far as we know the term "demigroupe" first appeared in the book "Eléments de la théorie des groupes abstraites", Paris 1904, by Monsieur l'Abbé J. A. de Séguier. In 1905 L. E. Dickson published an article "On semigroups and the general isomorphism between infinite groups" where he cites de Séguier. He defines semigroups explicitly as cancellative, associative structures which in his context naturally appear in a corrected version of a theorem of de Séguier. This event clearly marks a point where it becomes necessary to consider general transformations of a set rather than only permutations. In 1916 O. J. Schmidt introduces the term semigroup in his book "Abstract Group Theory" (in Russian) and means semigroups which are cancellative from both sides and possibly without identity. After that again more than ten years passed before semigroups became a direct object of investigation. In the first papers various different names were used like "group", "kernelgroup", "abstract composition system", "Übergruppe", " Schief", "abstract transmutation system", "Mischgruppe", cf. [Kna80]. Today we can say that the beginning of the theory of semigroups is marked by A. K. Suškewič [Sus28], D. Rees [Ree40], and P. Dubreil [Dub41].

In some respects the theory of semigroups has very close relations with group theory and in some other respects with ring theory. It is quite understandable that the early major contributions to the theory were strongly motivated by comparisons with groups and with rings. In more recent years the subject has developed its own characteristic problems, methods and results.

There exist many books on semigroup theory, we mention the works by A. K. Suškewič [SUS37], E. S. Ljapin [LJA60], A. H. Clifford and G. B. Preston [CL/PR61], [CL/PR67], L. Redei [RED63], M. Petrich [PET67], [PET73], [PET77], [PET84], T. Tamara [TAM72], M. Yamada [YAM76], J. M. Howie [HOW76], [HOW95], G. Lallement [LAL79], H. Jürgensen, F. Migliorini and J. Szép [JU/MI/SZ91], P. M. Higgins [HIG92], and P. A. Grillet [GRI95].

There exist many more which treat semigroup theory under different aspects as topology by A. B. Paalman-de Miranda [PAA64], K. H. Hofman and P. S. Mostert [HO/MO66], A. C. Shershin [SHE79], J. H. Carruth, J. A. Hildebrandt and R. J. Koch [CA/HI/KO83], [CA/HI/KO86], universal algebra by J. Almeida [ALM94], lattice theory by L. N. Shevrin and A. J. Ovsyannikov [SH/OV96], order theory by T. S. Blyth and M. F. Janowitz [BL/JA72], formal languages and automata theory by M. A. Arbib [ARB68], A. Ginzburg [GIN69], P. Deussen [DEU71], S. Eilenberg [EIL74], [EIL76], L. Budach and H.-J. Hoehnke [BU/HO75], S. Ginsberg [GIN75], J.-E. Pin [PIN86], V. N. Salij [SAL88], J. M. Howie [HOW91], H. J. Shyr [SHY91], or together with other algebraic structures by R. Bruck [BRU58], A. Pultr and V. Trnkova [PU/TR80], M. S. Putcha [PUT88], J. Okninski [OKN91], J. D. P. Meldrum [MEL95] and on the rather different approach from functional analysis by B. Hille [HIL48], B. Hille and R. S. Philipps [HI/PH57].

There is an enormous number of places in textbooks and monographs on groups, rings, algebraic systems, universal algebra, and lattice theory, where material and references on semigroups and their representations can be found.

Complementary to the monographs devoted to semigroup theory we present a systematic exposition of representation theory of monoids focussing on homological classification of monoids. We lead the reader to the frontiers of present research in this direction. At the same time we also look at familiar material from this perspective. We remark that one of the starting points of the book was a common survey article together with L. A. Skornjakov in 1975 [KI/KN/MI/SK75].

Recall that *group actions* (i.e. acts over groups) have been considered as long as groups have been known starting from P. Ruffini, A. Cayley, S. Lie [Lie91]. Even at the basic level of group theory in many university textbooks (S. Lang, A. I. Kostrikin, P. M. Cohn, I. N. Herstein, M. Artin etc.) we find effective and elegant ways of proving Sylow's theorems and other results on representation theory of finite groups applying results on orbits of group actions. Representations of semigroups by transformations of a set, i.e. acts over semigroups, play an essential role in semigroup theory from the beginnings (as can be seen from the title of A. K. Suschkewitsch's dissertation "The theory of action as generalized group theory"(in Russian), 1922).

A *representation of a semigroup S by transformations of a set* defines an S-act just as a representation of a ring R by endomorphisms of an abelian group defines an R-module. More precisely, a right act A_S over the monoid S is a set A for which a "product" $as \in A$ for $s \in S$ and $a \in A$ is defined such that for all $s_1, s_2 \in S$, $a \in A$

$$a(s_1 s_2) = (as_1)s_2 \quad \text{and} \quad a1 = a \text{ if } S \text{ has an identity } 1.$$

Left S-acts are defined analogously. Probably first, in this form, the definition of S-acts appeared in two papers by H.-J. Hoehnke [Hoe63], [Hoe66] with a different name in connection with the consideration of radicals of semigroups.

Note that in different books and articles we meet S-acts under different names: *S-automata, S-operands, S-polygons, S-sets, S-systems, monars, ac-*

tions, representation spaces, transition systems etc. In this book we use the name "S-act" to stress the role of the action of S on the set A.

Acts over semigroups appeared and were used in a variety of applications like algebraic automata theory, mathematical linguistics etc.

Semigroups and acts over them are the simplest type of algebras to which the *methods of universal algebra* must be applied, because their congruences are not defined by special subsets. Recall that for groups and rings the consideration of normal subgroups and ideals, respectively, avoids the direct use of congruences.

We note the remarkable *ubiquity of acts over semigroups* in mathematics and its applications. The starting points inside the theory of semigroups were results on representations by means of partial transformations of sets.

Besides of the type of representations considered in this book one has the classical Rees representations by matrices, Schützenberger representations by matrices which have at most one non-vanishing entry in each row or more generally by representations of monomial matrices over a semigroup. The extension of group representations to semigroups has been considered by R. R. Stoll [Sto44].

An approach to the subject was presented by E. J. Tully in his dissertation "Representation of a semigroup by transformations of sets", Tulane University, 1960, and [Tul61], and by H. J. Hoehnke [Hoe63], [Hoe65]. Earlier the theory of representations of inverse semigroups by one-to-one partial transformations was initiated by V. V. Wagner [Wag52], G. B. Preston [Pre54], and B. M. Schein [Sch61].

As we have mentioned earlier, acts were useful in the *structure theory of semigroups* when considering radicals of semigroups. The same is true for semigroups of quotients and torsion theories. On S-acts in connection with semigroups of quotients the reader may consult M. Delorme [Del69], M. D. Allouch [All74], [All76] and also H. J. Weinert's survey [Wei80]. Torsion theory for acts over groupoids was laid down by W. Lex and R. Wiegandt [Le/Wi81].

Aspects of *topological acts* over topological semigroups can be found in Isbell [Isb71], in [Kn/Mi73] and in [Nor94].

Partially ordered acts naturally appear in results on mappings between posets (see, for example, [HIG92], [BL/JA72], [Wag56] and [Kn/Mi73]). Injective ordered acts were touched by L. A. Skornjakov [Sko86a]. Lattice-ordered acts over bands and over monoids are used for the algebraic description of Petri nets by E. Wilkeit in [WIL98].

The investigation of *stochaastic acts* has been started by L. A. Skornjakov, *fuzzy acts* have been considered by V. Neklioudova [Nek97b].

In ring theory it is nowadays impossible to imagine the main directions without homological methods using categories of modules (see, for example, the book by R. Wisbauer [WIS91]). It is similarly important for monoids to consider associated categories of acts. Since these categories are highly non-additive (and generally they are as bad as the category of all sets; nevertheless the category of S-acts for a given S is a topos), one has to develope appropriate methods of homological algebra. This approach has proved to be a very effective tool in developing the so-called *homological classification of monoids.* The whole project of such classification for monoids, as also for rings, was suggested by L. A. Skornjakov ([Sko69a],[Sko69b]). We recall some pioneer results in the directions of:

- projective and injective acts, projective covers, injective envelopes (P. Berthiaume [Ber67], J. Isbell [Isb71]; C. S. Johnson and F. R. McMorris [Jo/Mc72b], U. Knauer [Kna72a], [Kna72b], and B. M. Schein [Sch76]);
- monoids over which all acts are (injective) projective (L. A. Skornjakov [Sko69b], E. H. Feller and R. Gantos [Fe/Ga69a], [Fe/Ga69b], [Fe/Ga69c]).
- Morita equivalences of monoids (B. Banaschewski [Ban72], U. Knauer [Kna72a], [Kna72b]);
- an abstract characterization of act categories (L. A. Skornjakov [Sko69a], see also [Ban72]);
- the tensor product for acts over monoids and flat acts (M. Delorme [Del69]), B. Stenström [Ste70], M. Kilp [Kil70], [KIL71].

Similarly, homological classification of Ω-rings (generalizing rings, semigroups, distributive lattices, semirings) was investigated by V. Fleischer [Fle75b].

Together with a right act A_S over a monoid S it is natural to consider its *endomorphism monoid* $E = \text{End}(A_S)$. Then ${}_EA_S$ is a biact, i.e. a right S-act, a left E-act such that $\sigma(aS) = \sigma(a)S$ for all $s \in S$, $a \in A$, $\sigma \in E$. We mention some directions of research citing early publications on:

- determinability of a free act by its endomorphism monoid (U. Knauer and A. V. Mikhalev [Kn/Mi73], V. Fleischer [Fle74], [Fle75a]);
- isomorphisms and antiisomorphisms of endomorphism monoids of generators (U. Knauer and A. V. Mikhalev [Kn/Mi92]);
- abstract characterization of endomorphism monoids of free acts (V. Fleischer [Fle74], [Fle75a]) and of simple acts (W. A. Lampe and W. Taylor [La/Ta82]);
- presentation of endomorphism monoids of free and projective acts as wreath products (L. A. Skornjakov [Sko79], U. Knauer and A. V. Mikhalev [Kn/Mi80a], [Kn/Mi80b], [Kn/Mi80c], and V. Fleischer and U. Knauer [Fl/Kn88]).

For acts over monoids in many cases one tries to model the situation in modules. At the same time in rings and modules representation theory of semigroups find applications, in particular, when considering endomorphism monoids of modules (see, for example, L. M. Gluskin [Glu66], A. V. Mikhalev [Mik66], [Mik96]) and when studying the multiplicative structure of rings (see, for example, B. M. Schein [Sch66], F. Eckstein [Eck69], C. F. Nelius [NEL74], A. V. Mikhalev [Mik88], K. I. Beidar, U. Knauer, A. V. Mikhalev [Be/Kn/Mi94]).

Universal algebra provides one of the ways of finding a common language for different branches of algebra. With any universal algebra A we may associate various derived semigroups S (for example, the group of all automorphisms, the monoid of all endomorphisms, the inverse semigroup of all partial automorphisms) and we get important acts over S.

We use *category theory* to study categories of acts over monoids, considering them as a non-additive analogue of module categories over rings.

On the other hand we may consider the category of S-acts as the category of functors from a one object category into the category of sets (similarly, the category of R-modules can be considered as functors from a one object category into the category of abelian groups). This point of view leads to the consideration of arbitrary small categories instead of one object categories (see L. V. Polin [Pol74b], [Pol74c], [Pol78]; E. V. Kacov [Kac76], [Kac78]; B. Elkins and J. A. Zilber [El/Zi76]). Another non-additive categorical generalization, connected with the tensor product functor, was suggested by B. Pareigis [PAR70], [Par77a], [Par77b], [Par78], [Par79], [Par80].

Acts over monoids and in particular over groups are appearing frequently in *combinatorial problems* (for example, in connection with the Burnside problem and with MacMahon's master theorem, see G. Lallement [LAL79]).

The model theory of modules has been investigated and continues to be extensively investigated (see M. Prest [PRE88], C. U. Jensen and H. Lenzing [JE/LE89], K. I. Beidar, A. V. Mikhalev, and G. E. Puniski [Be/Mi/Pu95]). In contrast, less is known about the *model theory* of acts over monoids. We mention some results on axiomatizability problems for some classes of acts: injective acts (L. A. Skornjakov [Sko78]), weakly injective acts (V. A. Kuzicheva, [Kuz80]), projective and flat acts (V. Gould [Gou87a]), regular acts (A. A. Stepanova [Ste91], [Ste94], E. V. Ovchinnikova [Ovc95]). Stability of acts has been discussed by T. G. Mustafin [Mus88], [Mus89], J. Fountain and V. Gould (unpublished), model companions of acts by V. Gould [Gou87b]; categoricity of the theory of acts by E. B. Kacov [Kac82],[Kac87].

The most notable difference between the two model theories over rings and over monoids is possibly the fact, discovered by T. G. Mustafin, J. Fountain, and V. Gould, that there exist S-acts which have unstable theories whereas all complete theories of modules are stable. Note that in this direction general results on perfect and coherent monoids and on finitely presented acts are especially useful.

For many model theory considerations with acts over monoids (as for modules over rings) it is convenient to have in mind so-called *multisorted systems* (see, for example G. Birkhoff and J. D. Lipson [Bi/Li70], A. V. Mikhalev [Mik86] and E. B. Kacov [Kac97a]). Note that an attempt to expose algebraic automata theory with accents on multisorted algebras and category theory was made by G. M. Brodskii [BRO88].

Finally we briefly touch relations of acts over monoids to other *mathematical applications*, in particular in graph theory, algebraic automata theory, theory of machines, theory of formal languages, mathematical linguistic, system theory, information theory, theory of communications and electronic circuits, data bases and other fields of theoretical computer science.

Graphs and diagrams provide an interesting intuitive guide on some results. If we have a graph $G = (V, E)$, where V is a non-empty set and $E \subseteq V \times V$ is a binary relation on V, then we may consider the endomorpism monoid $\mathrm{End}(G)$ of all endomorphisms of G, its submonoid $\mathrm{Send}(G)$ of strong endomorphisms, and the group $\mathrm{Aut}(G)$ of all automorphisms of G. In this way each graph G becomes a left act over any of these monoids. Several aspects of this view are spread in the the text of this book as applications, examples or exercises, where we usually give reference to the original papers.

The general idea of an automaton, which goes back to Descartes, Turing, Kleene, in spite of its simplicity, has led to deep results both in mathematics itself and in a wide spectrum of applications including theoretical computer science.

All forms in which one meets automata or machines have a set of states each of which can be transformed into other states by inputs. This explains the close relationship between automata and actions of semigroups. In fact, acts over free monoids are automata without outputs. We have many important examples of different type actions of monoids in algebraic automata theory. For example, X^*, the free monoid on the alphabet X, acts on the set of its subsets: $Yu = u^{-1}Y = \{s \in X^* \mid us \in Y\}$ for $Y \subseteq X^*$, $u \in X^*$.

A coherent mathematical presentation of algebraic automata theory was given by S. Eilenberg ([EIL74], [EIL76]). An introduction with many worked examples to this theory can be found in W. M. L. Holcombe [HOL82]. Other works to be mentioned here are the books by J. M. Howie [HOW91], G. Lallement [LAL79], J. D. P. Meldrum [MEL95], and J.-E. Pin [PIN86].

For automata theory *wreath product constructions* have great importance. Note that wreath product constructions for groups go back to Krasner–Kaloujnine's embedding theorem ([Kr/Ka50]) and to W. Specht [Spe33]. The idea of transferring embedding theorems for semigroups motivated B. H. Neuman also to consider wreath products for semigroups ([Neu60]). J. D. P. Meldrum [MEL95] devoted a book to wreath products of groups and semigroups. Group actions and wreath products are used by Kerber [Ker97] to describe combinatorial problems connected with graphs, codes, designs and chemical isomeres.

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