Faces of Mathematics

An Introductory Course for College Students

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Faces of Mathematics

THE THOMAS Y. CROWELL SERIES IN MATHEMATICS

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Our art work was done by Stan Olson, a young man with a knack for catching the nubbin of an idea and coming up with appropriate sketches. The entire manuscript was typed by Idella Varberg, who demonstrated that some marriages can withstand, in a spirit of continuing good humor, the strain of working closely on a project such as the present undertaking. We also wish to acknowledge the work of Penelope Behn and others in the production and design of this book.

These are the people with whom we have worked on a personal basis. There is another group, however, to whom we are indebted even though we have not known them personally. As explained in the Preface, we have made use of a good many puzzle problems in our book. Some of these problems are a well-established part of mathematical lore; their origin has long since been lost. Most of them appear in one guise or another in popular books of problems. We make no effort to identify the sources of problems we have used; they have found their way into our notes over many years of teaching. However, we are pleased to identify several puzzle books that have been favorites of ours. If we have succeeded in what we tried to do, then these books may well include some of our readers among the audiences they have served so well.

H. E. Dudeney, *Amusements in Mathematics*, Dover, New York, 1970. (reprint of a book first published in 1917)

H. E. Dudeney, *536 Puzzles and Curious Problems*, Scribners, New York, 1967.

E. R. Emmet, *Puzzles for Pleasure*, Emerson Books, Buchanan, N.Y., 1972.

Martin Gardner, *Mathematical Puzzles*, Crowell, New York, 1961. (see also, Gardner's regular monthly column in *Scientific American*)

J. F. Hurley, *Litton's Problematical Recreations*, Van Nostrand, New York, 1971.

B. A. Kordemsky, *The Moscow Puzzles*, Scribners, New York, 1972.

C. F. Linn, *Puzzles, Patterns and Pastimes*, Doubleday, Garden City, N.Y., 1969.

Preface

Without mathematics one will never penetrate to the depths of philosophy.
Without philosophy one will never penetrate to the depths of mathematics.
Without both one will never penetrate to the depths of anything.
Leibniz

It has long been held that anyone who aspires to be educated must study mathematics. We still believe it, and this book is intended to be a source book for those who want to see what mathematics can contribute to a liberal education. In particular, we have in mind those college students who plan to take just one or two semesters of mathematics. Perhaps they want to satisfy a distributive requirement, or perhaps they are prospective elementary school teachers who need a broadened and deepened perspective on mathematics.

A number of books have addressed themselves to this audience. We think they generally miss the mark for either of two reasons. Some try to survey the content of mathematics, offering a smorgasbord from which users may choose according to their taste. Such books are often superficial, although this is not our principal objection. The availability of interesting and potentially practical topics is not the only reason—or perhaps even the main reason—great thinkers insisted that educated people should study mathematics. They believed, as we believe, that the study of mathematics can help us to learn something about thinking itself: how to state our problems clearly, sort out the relevant from the irrelevant, argue coherently, and abstract some common properties from many individual situations. It is toward these goals that we wish to move.

This brings us to the second kind of book available for the purposes we have in mind. This type of book emphasizes the methodology rather than the content of mathematics. Attention is focused on how we think, rigor and clarity, common methods of proof, the way in which great mathematical ideas have developed, and the foundations of mathematics. We have been greatly impressed with many of these books—in particular, mention should be made of the influence of such writers as Polya, Wilder, and Richardson. Books of this type, however, have one drawback: they are too difficult for the audience we have in mind.

We have tried to steer a middle course. Insofar as it was consistent with maintaining a light, readable, often humorous style that would appeal to our audience, we have selected topics that can be presented in some depth. Moreover, we have continually addressed ourselves to the larger contention that mathematics is the ideal arena in which to develop skill in the areas of information organization, problem analysis, and argument presentation.

It is our belief, not shared (we are sad to say) by all educators, that a course developed along the lines of this book would be an excellent preparation for an elementary school teacher. We feel that our text, in its emphasis on lively problems and its attention to those mathematical concepts judged to be essential knowledge for all educated people, offers an attractive alternative to the dreary routine involving sets, distinguishing between numbers and numerals, and the associative law of addition—the usual fare in texts designed for teachers. The necessary material about number systems should, of course, be included in such a course.

A WORD ABOUT THE TITLE

We chose the title *Faces of Mathematics* for two reasons. First, we wanted to emphasize the fact that mathematics was developed by human beings, real people with real faces. True, they may have had special talents, but on the whole they lived their lives subject to the same constraints as anyone else. Results in mathematics do not arise through divine revelation; they represent the hard work of individual men and women. The faces and brief biographies of many of the most significant contributors to this field appear on the following pages.

Second, we wanted to suggest the analogy that mathematics is like a finely cut diamond; it must be seen from several sides to be fully appreciated. Each view exposes a new face with its own distinctive features. Four of these faces—solving problems, finding order, building models, and creating abstractions—reflect those activities most characteristic of mathematicians. We have organized our book around these four faces.

A SPECIAL WORD TO STUDENTS

Many years of teaching have convinced us that most students who fall within this book's intended audience approach mathematics with fear and trembling. We have made every effort to ease this anxiety by using simple examples, clear explanations, and a limited technical vocabulary. Our aim is to demonstrate that mathematics is interesting, relevant, and learnable.

We believe that problem solving is the heart of mathematics.

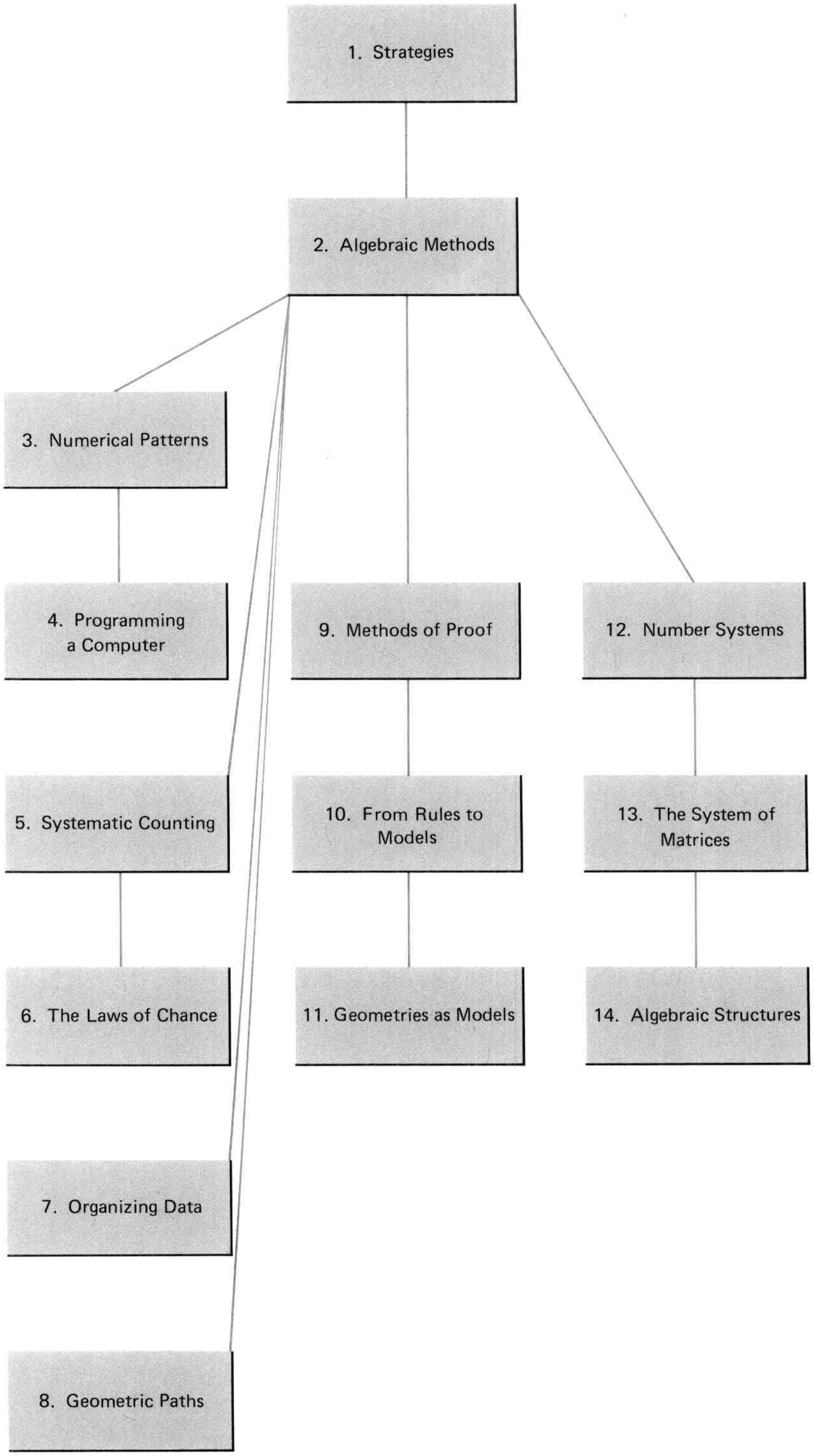
The great mathematicians were problem solvers. Every person, be he mathematician, painter, scientist, or carpenter, must solve problems; it is part of being alive. For this reason, we begin our book with strategies for problem solving. Problems, moreover, are the unifying thread that holds our book together. Most sections begin with a problem; every section ends with a host of problems for you to try. They are carefully arranged in order of increasing difficulty, the most challenging being identified with an asterisk. Be sure to work at the problems; it is the only way to learn mathematics. It is also the activity most likely to help you later in life.

ADVICE TO TEACHERS

This text can be used in a variety of ways. The book contains sufficient material for a full-year (two-semester) course. It is also easy to make selections for the typical semester course offered at many colleges. Both of us have used a preliminary version of the book in one-semester courses. Professor Varberg's course, which emphasized problem solving, was based on Chapters 1, 2, 3, 4, 7, 8, and 12. Professor Roberts's course was more philosophical, with special attention given to clear thinking and precise writing. He used most of Chapters 1, 2, 9, 10, 11, 12, 13, and 14. There are many other possibilities. The Dependence Chart will help you design a course to your liking.

Experience suggests that students profit from an early review of the appendixes, especially A and B. These appendixes are designed to refresh students' memories about things learned long ago but possibly forgotten. An appropriate time to consider them is right after Chapter 1.

Dependence Chart



Contents

Faces of Mathematics

A great discovery solves a great problem but there is a grain of discovery in the solution of any problem. Your problem may be modest; but if it brings into play your inventive faculties, and if you solve it by your own means, you may experience the tension and enjoy the triumph of discovery.
George Polya

Part I
Solving Problems

PROBLEMS

There was a time when almost everyone associated elementary mathematics with long lists of problems to be solved: theoretical problems and practical problems, problems requiring long computations and problems beautiful in their simplicity, many monotonously simple drill problems and a few utterly baffling problems. To some, mathematics seemed nothing more than a collection of memorized tricks which could, with luck, be matched to the problems they were designed to solve.

In an effort to get away from this view, the so-called new math was developed to emphasize the structure and unity of mathematics. The intention was laudable, and in certain ways successful, but when pushed too far, this approach too became pedantic. One feels the need to learn abstract principles only when one has worked on concrete problems. Skeletons are wonderfully useful, but it is easier to sell pictures of those that are covered with meat. Problems are the meat of mathematics and the focus of this book.

Every subject has its problems. In contrast, however, to many useful areas of human inquiry (medicine, psychology, economics, etc.) where a clear and enduring answer is seldom expected, mathematical problems admit the possibility of uncontestably correct answers. They therefore afford us an excellent medium in which we can focus attention not on the answers but on how they are obtained. In Part I, we undertake such a study, suggesting that there are principles applicable to solving a host of common problems.

It is not essential to our purposes to consider only practical problems. What is essential is that our problems illustrate the principles we have in mind, that they be interesting, that they pose a challenge—yet seem enough within grasp to be tantalizing—and that they draw out from our imagination creative ideas about which we are pleased to say, "I thought of that."

George Polya (1887–   )

George Polya was born in Hungary and educated at the universities of Budapest, Vienna, Göttingen, and Paris. After teaching for 26 years at the Swiss Federal Institute of Technology in Zürich, he became affiliated with Stanford University where he has continued his research in advanced mathematics, research that has resulted in over 200 papers and several books.

Polya's research in pure mathematics has earned him a place of honor among the world's leading contemporary mathematicians. But he is also famous for the research and writing he has done on the nature of problem solving. His books, *How to Solve It,* 2nd ed. (Garden City, N.Y.: Doubleday, 1959), *Mathematics and Plausible Reasoning* (2 vols.) (Princeton, N.J.: Princeton University Press, 1954), and *Mathematical Discovery* (2 vols.) (New York: Wiley, 1962) are widely read expositions of the art of solving problems. We are pleased to acknowledge that the ideas we express in Part I have been profoundly influenced by reading Polya's books.

Chapter 1
Strategies

Solving a problem is similar to building a house. We must collect the right material, but collecting the material is not enough; a heap of stones is not yet a house. To construct the house or the solution, we must put together the parts and organize them into a purposeful whole.

GEORGE POLYA