# College Calculus with Analytic Geometry

THIRD EDITION

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MURRAY H. PROTTER

CHARLES B. MORREY, JR.

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*University of California, Berkeley*

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# PREFACE

*College Calculus with Analytic Geometry* is designed to meet the needs of a majority of students taking a three-semester course in calculus at a college or university. The text leans heavily on the intuitive approach, gives many illustrative examples, emphasizes applications wherever suitable, and has a large selection of graded exercises. The changes in this third edition reflect the many suggestions we have received from teachers and students who have used the previous editions.

In Chapter 1 we discuss inequalities, with emphasis on the use of the absolute-value symbol. Set notation, an important concept for students, is introduced in Chapter 2; we use it in the text in only those cases in which ambiguity could result from the employment of classical notation. However, when set notation is cumbersome and does not contribute to the understanding of the subject matter, we continue to use standard notation.

Chapter 4 contains an intuitive development of limits, a geometric interpretation of derivative, and a discussion of continuity and limits of sequences. This material prepares the student for the thorough treatment of the differentiation of algebraic functions and the Chain Rule in Chapter 5, as well as for the applications to problems of maxima and minima, related rates, and approximation in Chapter 6. The section on differential notation in Chapter 6 has been rewritten and simplified.

Chapter 7 begins with a careful treatment of area (Jordan content), an approach which is advantageous in the development of the definition of the integral. We establish various basic properties of the integral, such as the Theorem of the Mean and two forms of the Fundamental Theorem of Calculus. There are also applications to problems of fluid pressure and work, but these may be omitted without loss of continuity in the presentation.

In Chapter 8 we resume the work on analytic geometry which we began in Chapter 3, now using the tools of calculus in conjunction with the geometric theory. In order to achieve flexibility in the use of the text, we have placed the more-detailed topics of analytic geometry in a special appendix. Since many entering students are well prepared in analytic geometry, the instructor may wish to cover the topics in Chapters 3 and 8 very quickly, treating them as a review. On the other hand, he or she could choose to make a thorough presentation of the subject by assigning both Chapters 3 and 8 as well as the appendix on analytic geometry. The total amount of material on conics and related subjects is at least as great in this book as that found in most texts devoted to analytic geometry alone.

We next define the natural logarithm by means of the integral, with the exponential function defined as its inverse. This topic, as well as the differentiation and integration of trigonometric functions and inverse trigonometric functions, is treated in Chapter 9. We have also provided an appendix with a brief review of trigonometry for those students who, in the interim since their study of trigonometry, may have forgotten the definitions of the elementary trigonometric functions, as well as the formulas and identities most frequently used in calculus.

In this edition we provide in Chapter 13 a unified development of vectors in two and three dimensions. In order to do this, we have placed the topics on three-dimensional analytic geometry (except for quadrics) in Chapter 12, so that the student can master the concepts of geometric objects in space before he studies vectors in three dimensions. The study of quadric surfaces, a rather special topic and one not required for vector theory, has been transferred to an appendix.

Chapters 14 and 15 discuss techniques in integration and their applications.

The study of infinite series, the subject of Chapter 16, completes the course in the calculus of functions of one variable. Chapters 17 and 18 are devoted to the initial topics in the calculus of functions of several variables. Partial differentiation, line integrals, and applications are taken up in Chapter 17. In this edition we mention only briefly the various symbols for partial differentiation and use more-uniform notation than we did in our previous editions. A definition of volume (Jordan content), analogous to that given for area in Chapter 7, is discussed in Chapter 18. We also cover the elements of multiple integration, with applications to area, volume, and mass.

Besides those already mentioned, three other useful appendixes are provided. The one on hyperbolic functions can be used by those students in engineering and technology who are most likely to need such functions for later courses. There is an appendix on the axioms of algebra which should provide interesting and useful additional reading for serious students interested in continuing in mathematics. We have also provided a table of indefinite integrals and a brief description on how it is to be used.

An important feature of this edition is the addition of a large number of challenging exercises which have been inserted at the end of almost every section. These exercises, together with the routine exercises and those retained from the second edition, should give the instructor the flexibility he or she needs to assign large numbers of exercises of moderate difficulty for the average student, as well as a substantial number of challenging exercises for the more capable student.

Because of the widespread use of the metric system in science and technology and the gradual conversion to this system in industry and commerce, we have employed metric units consistently in those problems and applications which involve measurement.

*Berkeley, California*
*January 1977*

M. H. P.
C. B. M., Jr.

# CONTENTS

# 1

# INEQUALITIES

## 1. INEQUALITIES*

Almost all high school students learn plane geometry as a single logical development in which theorems are proved on the basis of a system of axioms or postulates. Unlike plane geometry, however, algebra has traditionally been taught in high school without the aid of a formal logical system. In this method the student simply learns a few rules—or many—for manipulating algebraic quantities; these rules lead to success in solving problems but do not shed any light on the structure of algebra.

The usual rules of algebra are logical consequences of the system of axioms known as the Axioms of Algebra. To prove the rules we use for manipulating algebraic expressions directly from the Axioms, as in Euclidean geometry, would be cumbersome and unwieldy. Therefore we shall assume that the reader is familiar with the usual laws of algebra and begin with a discussion of inequalities. The Axioms of Algebra are given in Appendix 4 at the end of the book, and we recommend their study to students unfamiliar with them.

In elementary algebra and geometry we study equalities almost exclusively. The solution of linear and quadratic algebraic equations, the congruence of geometric figures, and relationships among various trigonometric functions are topics concerned with equality. As we progress in the development of mathematical ideas—especially in that branch of mathematics of which calculus is a part—we shall see that the study of inequalities is both interesting and useful. An inequality is involved when we are more concerned with the approximate size of a quantity than we are with its true value. Since the proofs of some of the most important theorems in calculus depend on certain approximations, it is essential that we develop a facility for working with inequalities.

We shall be concerned with inequalities among real numbers, and we begin by recalling some familiar relationships. Given that $a$ and $b$ are any two real numbers, the symbol

$$a < b$$

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This chapter and Chapter 2 consist of review material for many students of calculus. Students who do ot have a thorough working knowledge of inequalities should begin here. Readers familiar with equalities may start with Chapter 2.

means that $a$ **is less than** $b$.* We may also write the same inequality in the *opposite direction*,

$$b > a,$$

which is read $b$ **is greater than** $a$.

The rules for handling inequalities can be proved on the basis of the Axioms for Algebra. The rules themselves are only slightly more complicated than the ones we learned in algebra for equalities. However, the differences are so important that we state them as four Theorems about Inequalities and they should be studied carefully.

**Theorem 1.** *If $a < b$ and $b < c$, then $a < c$. In words: if $a$ is less than $b$ and $b$ is less than $c$, then $a$ is less than $c$.*

**Theorem 2.** *If $c$ is any number and $a < b$, then it is also true that $a + c < b + c$ and $a - c < b - c$. In words: if the same number is added to or subtracted from each side of an inequality, the result is an inequality in the same direction.*

**Theorem 3.** *If $a < b$ and $c < d$ then $a + c < b + d$. That is, inequalities in the same direction may be added.*

It is important to note that in general inequalities may not be subtracted. For example, $2 < 5$ and $1 < 7$. We can say, by addition, that $3 < 12$, but note that subtraction would state the absurdity that 1 is less than $-2$.

**Theorem 4.** *If $a < b$ and $c$ is any positive number, then*

$$ac < bc,$$

*while if $c$ is a negative number, then*

$$ac > bc.$$

*In words: multiplication of both sides of an inequality by the same positive number preserves the direction, while multiplication by a negative number reverses the direction of the inequality.*

Since dividing an inequality by a number $d$ is the same as multiplying it by $1/d$, we see that Theorem 4 applies for division as well as for multiplication.

From the geometric point of view we associate a horizontal axis with the totality of real numbers. The origin may be selected at any convenient point, with positive numbers to the right and negative numbers to the left (Fig. 1–1). For every real number there will be a corresponding point on the line and, conversely, every point will represent a real number. Then the inequality $a < b$ may be read: *$a$ is to the left of $b$*. This geometric way of looking at inequalities is frequently of

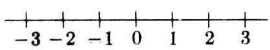


**Figure 1–1**

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* Which is true if and only if $b - a$ is positive (see Appendix 4, §2).


Figure 1–2


Figure 1–3

help in solving problems. It is also helpful to introduce the notion of an *interval of numbers* or *points*. If $a$ and $b$ are numbers (as shown in Fig. 1–2), then the **open interval** from $a$ to $b$ is the collection of all numbers which are both larger than $a$ and smaller than $b$. That is, an open interval consists of all numbers *between a and b*. A number $x$ is between $a$ and $b$ if *both* inequalities $a < x$ and $x < b$ are true. A compact way of writing this is

$$a < x < b.$$

The **closed interval** from $a$ to $b$ consists of all the points between $a$ and $b$, *including* $a$ and $b$ (Fig. 1–3). Suppose a number $x$ is either equal to $a$ or larger than $a$, but we don't know which. We write this conveniently as $x \geq a$, which is read: *x is greater than or equal to a*. Similarly, $x \leq b$ is read: *x is less than or equal to b*, and means that $x$ may be either smaller than $b$ or may be $b$ itself. A compact way of designating a closed interval from $a$ to $b$ is to state that it consists of all points $x$ such that

$$a \leq x \leq b.$$

An interval which contains the endpoint $b$ but not $a$ is said to be **half-open on the left.** That is, it consists of all points $x$ such that

$$a < x \leq b.$$

Similarly, an interval containing $a$ but not $b$ is called **half-open on the right,** and we write

$$a \leq x < b.$$

Parentheses and brackets are used as symbols for intervals in the following way:

$(a, b)$ for the open interval: $a < x < b$,
$[a, b]$ for the closed interval: $a \leq x \leq b$,
$(a, b]$ for the interval half-open on the left: $a < x \leq b$,
$[a, b)$ for the interval half-open on the right: $a \leq x < b$.

We can extend the idea of an interval of points to cover some unusual cases. Suppose we wish to consider *all* numbers larger than 7. This may be thought of as an interval extending to infinity on the right. (See Fig. 1–4.) Of course, *infinity is not a number*, but we use the symbol $(7, \infty)$ to represent all numbers larger than 7. We could also write: all numbers $x$ such that

$$7 < x < \infty.$$

In a similar way, the symbol $(-\infty, 12)$ will stand for all numbers less than 12. The double inequality

$$-\infty < x < 12$$

is an equivalent way of representing all numbers $x$ less than 12.

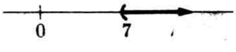


Figure 1–4

The first-degree equation $3x + 7 = 19$ has a unique solution, $x = 4$. The quadratic equation $x^2 - x - 2 = 0$ has two solutions, $x = -1$ and $x = 2$. The trigonometric equation $\sin x = \frac{1}{2}$ has an infinite number of solutions: $x = 30°$, $150°, 390°, 510°, \ldots$. *The solution of an inequality* involving a single unknown, say $x$, *is the collection of all numbers which make the inequality a true statement.* Sometimes this is called the **solution set.** For example, the inequality

$$3x - 7 < 8$$

has as its solution *all* numbers less than 5. To demonstrate this we argue in the following way. If $x$ is a number which satisfies the above inequality we can, by Theorem 2, add 7 to both sides of the inequality and obtain a true statement. That is, we have

$$3x - 7 + 7 < 8 + 7 \quad \text{or} \quad 3x < 15.$$

Now, dividing both sides by 3 (Theorem 4), we obtain

$$x < 5$$

and observe that *if* $x$ is a solution, *then* it is less than 5. Strictly speaking, however, we have not *proved* that every number which is less than 5 is a solution. In an actual proof we would begin by supposing that $x$ is any number less than 5; that is,

$$x < 5.$$

We multiply both sides by 3 (Theorem 4) and then subtract 7 (Theorem 2) to get

$$3x - 7 < 8,$$

the original inequality. Since the condition that $x$ is less than 5 implies the original inequality, we have proved the result. The important thing to notice is that the proof consisted of *reversing* the steps of the original argument which led to the solution $x < 5$ in the first place. So long as each of the steps we take is *reversible,* the above procedure is completely satisfactory so far as obtaining solutions is concerned. The step going from $3x - 7 < 8$ to $3x < 15$ is reversible, since these two inequalities are equivalent. Similarly, the inequalities $3x < 15$ and $x < 5$ are equivalent. Finally, note that the solution set consists of all numbers in the interval $(-\infty, 5)$.

Methods for the solution of various types of simple algebraic inequalities are shown in the following examples, which should be studied carefully.

**Example 1.** Solve for $x$:

$$-7 - 3x < 5x + 29.$$

*Solution.* Subtract $5x$ from both sides, getting

$$-7 - 8x < 29.$$

Multiply both sides by $-1$, reversing the direction of the inequality, to obtain

$$7 + 8x > -29.$$

Subtracting 7 from both sides yields $8x > -36$, and dividing by 8 gives the solution

$$x > -\tfrac{9}{2},$$

or, stated in interval form: all $x$ in the interval $(-\frac{9}{2}, \infty)$. ◀

To verify the correctness of the result, it is necessary to perform the above steps in reverse order. However, the observation that each individual step is reversible is sufficient to check the validity of the answer.

**Example 2.** Solve for $x$ $(x \neq 0)$:

$$\frac{3}{x} < 5.$$

*Solution.* We have an immediate inclination to multiply both sides by $x$. However, since we don't know in advance whether $x$ is positive or negative, we must proceed cautiously. We do this by considering two cases: (1) $x$ is positive, and (2) $x$ is negative.

CASE 1. Suppose $x > 0$. Then multiplying by $x$ preserves the direction of the inequality (Theorem 4), and we get

$$3 < 5x.$$

Dividing by 5, we find that $x > \frac{3}{5}$. This means that we must find all numbers which satisfy *both* of the inequalities

$$x > 0 \qquad \text{and} \qquad x > \tfrac{3}{5}.$$

Clearly, any number greater than $\frac{3}{5}$ is also positive, and the solution in Case 1 consists of all $x$ in the interval $(\frac{3}{5}, \infty)$.

CASE 2. $x < 0$. Multiplying by $x$ *reverses* the direction of the inequality. We have

$$3 > 5x,$$

and therefore $\frac{3}{5} > x$. We seek all numbers $x$, such that *both* of the inequalities

$$x < 0 \qquad \text{and} \qquad x < \tfrac{3}{5}$$

hold. The solution in Case 2 is the collection of all $x$ in the interval $(-\infty, 0)$. A way of combining the answers in the two cases is to state that the solution set consists of all numbers $x$ *not* in the closed interval $[0, \frac{3}{5}]$. (See Fig. 1–5.) ◀

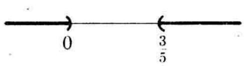


**Figure 1–5**

**Example 3.** Solve for $x$ $(x \neq -2)$:

$$\frac{2x - 3}{x + 2} < \frac{1}{3}.$$

*Solution.* As in Example 2, we must consider two cases, according to whether $x + 2$ is positive or negative.

CASE 1. $x + 2 > 0$. We multiply by $3(x + 2)$, which is positive, getting

$$6x - 9 < x + 2.$$

Adding $9 - x$ to both sides, we have

$$5x < 11, \qquad \text{from which} \qquad x < \tfrac{11}{5}.$$

Since we have already assumed that $x + 2 > 0$, and since we must have $x < \frac{11}{5}$, we see that $x$ must be larger than $-2$ and smaller than $\frac{11}{5}$. That is, the solution set consists of all $x$ in the interval $(-2, \frac{11}{5})$.

CASE 2. $x + 2 < 0$. Again multiplying by $3(x + 2)$ and *reversing* the inequality, we obtain

$$6x - 9 > x + 2,$$

or $5x > 11$ and $x > \frac{11}{5}$. In this case, $x$ must be less than $-2$ and greater than $\frac{11}{5}$, which is impossible. Combining the cases, we get as the solution set all numbers in $(-2, \frac{11}{5})$. See Fig. 1–6.

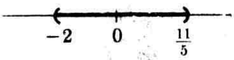


**Figure 1–6** ◀

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## PROBLEMS

In Problems 1 through 14, solve for $x$.

1. $2x - 3 < 7$
2. $2x + 4 < x - 5$
3. $5 - 3x < 14$
4. $2 - 5x < 3 + 4x$
5. $2(3x - 6) < 4 - (2 + 5x)$
6. $x - 4 < \dfrac{2x}{3} + \dfrac{2 - 3x}{5}$
7. $\dfrac{4}{x} < \dfrac{3}{5}$
8. $\dfrac{x - 1}{x} < 4$
9. $\dfrac{x + 3}{x - 2} < 5$
10. $\dfrac{2}{x} - 3 < \dfrac{4}{x} + 1$
11. $\dfrac{x + 3}{x - 4} < -2$
12. $\dfrac{2 - x}{x + 1} < \dfrac{3}{2}$
13. $\dfrac{x}{3 - x} < 2$
14. *$\dfrac{x - 2}{x + 3} < \dfrac{x + 1}{x}$

In Problems 15 through 23, find the values of $x$, if any, for which both inequalities hold.

15. $2x - 7 < 5 - x$ and $3 - 4x < \frac{5}{3}$
16. $3x - 8 < 5(2 - x)$ and $2(3x + 4) - 4x + 7 < 5 + x$
17. $\dfrac{2x+6}{3} - \dfrac{x}{4} < 5$ and $15 - 3x < 4 + 2x$
18. $3 - 6x < 2(x + 5)$ and $7(2 - x) < 3x + 8$

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* Starred problems are those that are unusually difficult.

*19. $\frac{2}{x-1} < 4$ and $\frac{3}{x-2} < 7$

*20. $\frac{x-2}{x+1} < 3$ and $\frac{3-x}{x-2} < 5$

*21. $\frac{3}{x} < 5$ and $x + 2 < 7 - 3x$

*22. $\frac{2}{x} < 1$ and $(x-2)(x+3) < 0$

*23. $\frac{3}{x} < 2$ and $(x-1)(x+4) < 0$

24. Show that Theorem 3 for inequalities may be derived from Theorems 1 and 2.

25. Given that $a$, $b$, $c$, and $d$ are all positive numbers, and that $a < b$ and $c < d$; show that $ac < bd$.

*26. a) State the most general circumstances in which the hypotheses $a < b$ and $c < d$ imply that $ac < bd$.
b) Given that $a < b$ and $c < d$, when is it true that $ac > bd$?

27. If $x$ is a positive number, prove that

$$x + \frac{1}{x} \geq 2.$$

28. a) If $x$ and $y$ are positive numbers, show that

$$\left(\frac{1}{x} + \frac{1}{y}\right)(x + y) \geq 4.$$

b) If $x$, $y$, and $z$ are positive numbers, show that

$$\left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z}\right)(x + y + z) \geq 9.$$

c) If $x$, $y$, $z$, and $w$ are positive numbers, show that

$$\left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z} + \frac{1}{w}\right)(x + y + z + w) \geq 16.$$

*d) Generalize the above results to $n$ numbers $x_1, x_2, \ldots, x_n$.

*29. If $x$ and $y$ are any numbers different from zero, show that

$$\frac{x^2}{y^2} + \frac{16y^2}{x^2} + 24 \geq \frac{8x}{y} + \frac{32y}{x}.$$

*30. Let $x$ and $y$ be positive numbers with $x \geq y$. Show that

$$\frac{x}{y} + 3\frac{y}{x} \geq \frac{y^2}{x^2} + 3.$$

Show that the inequality is reversed if $y \geq x$.

## 2. ABSOLUTE VALUE

If $a$ is any positive number, the **absolute value** of $a$ is defined to be $a$ itself. If $a$ is negative, the absolute value of $a$ is defined to be $-a$. The absolute value of zero is