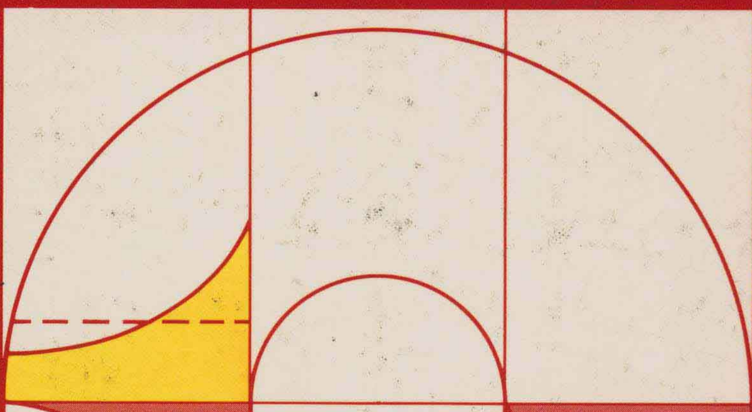
Phillip L. Gould

Introduction to Linear Elasticity



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Introduction to Linear Elasticity

To my children

Elizabeth Sue
Nathan Charles
Rebecca Blair
Joshua Robert

Preface

"Elasticity is one of the crowning achievements of Western culture!" exclaimed my usually reserved colleague Professor George Zahalak during a meeting to discuss the graduate program in Solid Mechanics. Although my thoughts on the theory of elasticity had not been expressed in such noble terms, it was the same admiration for the creative efforts of the premier physicists, mathematicians and mechanicians of the 19th and 20th century that led me to attempt to popularize the basis of solid mechanics in this introductory form.

The book is intended to provide a thorough grounding in tensor-based theory of elasticity, which is rigorous in treatment but limited in scope. It is directed to advanced undergraduate and graduate students in civil, mechanical or aeronautical engineering who may ultimately pursue more applied studies. It is also hoped that a few may be inspired to delve deeper into the vast literature on the subject. A one-term course based on this material may replace traditional Advanced Strength of Materials in the curriculum, since many of the fundamental topics grouped under that title are treated here, while those computational techniques that have become obsolete due to the availability of superior, computer-based numerical methods are omitted.

Little, if any, originality is claimed for this work other than the selection, organization and presentation of the material. The principal historical contributors are noted in the text and several modern references are liberally cited.

My personal interest in the theory of elasticity was kindled at Northwestern University through a course offered by Professor George Herrmann, now of Stanford University. I am also indebted to my colleague Professor S. Sridharan who has class-tested the text, pointed out errors and omissions, and contributed some challenging exercises, as well as to Mr. Moujalli Hourani for carefully reading the manuscript. I am grateful to Ms. Kathryn Schallert for typing the manuscript.

Phillip L. Gould

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CHAPTER 1

Introduction and Mathematical Preliminaries

1.1 Scope

The *theory of elasticity* comprises a consistent set of equations which uniquely describe the state of stress, strain and displacement at each point within an elastic deformable body. Solutions of these equations fall into the realm of *applied mathematics*, while applications of such solutions are of *engineering* interest. When elasticity is selected as the basis for an engineering solution, a rigor is accepted that distinguishes this approach from the alternatives, which are mainly the based on the strength of materials with its various specialized derivatives such as the theories of rods, beams, plates and shells. The distinguishing feature between the various alternative approaches and the theory of elasticity is the *pointwise* description embodied in elasticity, without resort to expedients such as Navier's hypothesis of plane sections remaining plane.

The theory of elasticity contains *equilibrium* equations relating the stresses; *kinematic* equations relating the strains and displacements; *constitutive* equations relating the stresses and strains; *boundary conditions* relating to the physical domain; and *uniqueness* constraints relating to the applicability of the solution. Origination of the theory of elasticity is attributed to Louis–Marie–Henri Navier, Simon–Denis Poisson and George Green in the first half of the 19th century [1.1].

In subsequent chapters, each component of the theory will be developed in full from the fundamental principles of physics and mathematics. Some limited applications will then be presented to illustrate the potency of the theory as well as its limitations.

1.2 Vector Algebra

A *vector* is a directed line segment in the physical sense. Referred to the unit basis vectors $(\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z)$ in the Cartesian coordinate system (x, y, z), an arbitrary vector $\mathbf{A}$ may be written in component form as

$$\mathbf{A} = A_x\mathbf{e}_x + A_y\mathbf{e}_y + A_z\mathbf{e}_z. \tag{1-1}$$

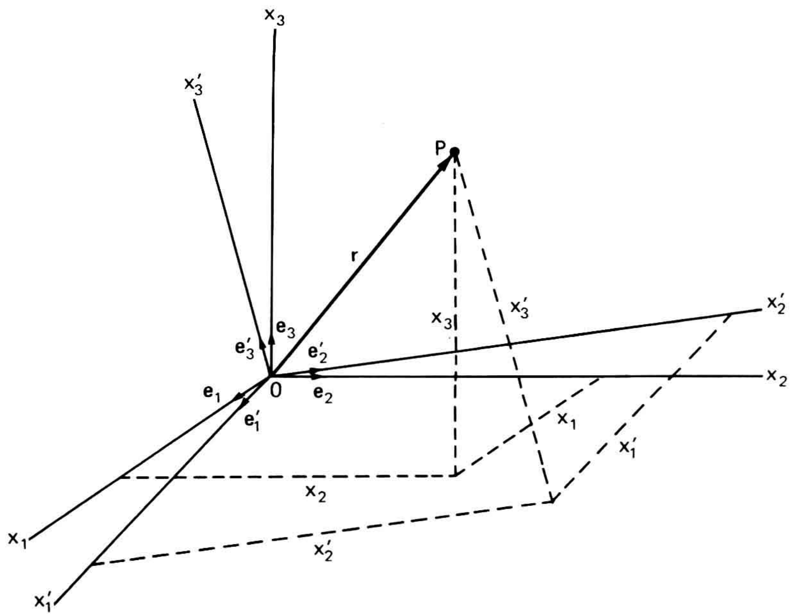

Fig. 1-1 Cartesian coordinate systems (after Tauchert, *Energy Principles in Structural Mechanics*, McGraw–Hill, 1974). Reproduced by permission.

Alternately, the Cartesian system could be numerically designated as (x_1, x_2, x_3), whereupon

$$\mathbf{A} = A_1\mathbf{e}_1 + A_2\mathbf{e}_2 + A_3\mathbf{e}_3. \tag{1-2}$$

The latter form is common in elasticity. An example is vector $\mathbf{r}$ in Fig. 1-1, where the unit vectors $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$ are identified.

Beyond the physical representation, it is often sufficient to deal with the components alone as ordered triples,

$$\mathbf{A} = (A_1, A_2, A_3). \tag{1-3}$$

The length or magnitude of $\mathbf{A}$ is given by

$$|\mathbf{A}| = \sqrt{A_1^2 + A_2^2 + A_3^2}. \tag{1-4}$$

Vector equality, addition and subtraction are trivial. Vector multiplication has two forms. The *inner*, *dot*, or *scalar* product is

$$\begin{aligned} C &= \mathbf{A}\cdot\mathbf{B} \\ &= A_1B_1 + A_2B_2 + A_3B_3 \\ &= |\mathbf{A}|\,|\mathbf{B}|\cos\theta_{AB}. \end{aligned} \tag{1-5}$$

Additionally, there is the *outer*, *cross* or *vector* product

$$\begin{aligned} \mathbf{C} &= \mathbf{A} \times \mathbf{B} \\ &= (A_2B_3 - A_3B_2)\mathbf{e}_1 + (A_3B_1 - A_1B_3)\mathbf{e}_2 + (A_1B_2 - A_2B_1)\mathbf{e}_3, \end{aligned} \tag{1-6}$$

which is conveniently evaluated as a determinant

$$\mathbf{C} = \mathbf{A} \times \mathbf{B} = \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \end{vmatrix};$$

C is perpendicular to the plane containing **A** and **B**.

1.3 Scalar and Vector Fields

1.3.1 Definitions: A *scalar* quantity expressed as a function of the Cartesian coordinates such as

$$f(x_1, x_2, x_3) = \text{constant} \tag{1-7}$$

is known as a *scalar field*. An example is the temperature at a point.

A *vector* quantity similarly expressed, such as $\mathbf{A}(x_1, x_2, x_3)$, is called a *vector field*. An example is the velocity of a particle. We are concerned with changes or derivatives of these fields.

1.3.2 Gradient: The *gradient* of a scalar field f is defined as

$$\begin{aligned} \mathbf{grad}\, f &= \nabla f \\ &= \frac{\partial f}{\partial x_1}\mathbf{e}_1 + \frac{\partial f}{\partial x_2}\mathbf{e}_2 + \frac{\partial f}{\partial x_3}\mathbf{e}_3 \\ &= \left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \frac{\partial f}{\partial x_3}\right), \end{aligned} \tag{1-8}$$

grad f is a *vector* point function which is orthogonal to the surface $f =$ constant, everywhere. Conversely, the components of **grad** f may be found by the appropriate dot product, for example,

$$\frac{\partial f}{\partial x_1} = \mathbf{e}_1 \cdot \nabla f \tag{1-9}$$

The del operator ∇ may be treated as a vector

$$\nabla(\;) = \frac{\partial(\;)}{\partial x_1}\mathbf{e}_1 + \frac{\partial(\;)}{\partial x_2}\mathbf{e}_2 + \frac{\partial(\;)}{\partial x_3}\mathbf{e}_3, \tag{1-10}$$

which is mathematically convenient.

1.3.3 Divergence: The *divergence* of a vector field **A** is defined as

$$\begin{aligned}\operatorname{div}\mathbf{A} &= \nabla\cdot\mathbf{A}\\ &= \frac{\partial A_1}{\partial x_1}+\frac{\partial A_2}{\partial x_2}+\frac{\partial A_3}{\partial x_3},\end{aligned}\qquad (1\text{-}11)$$

which is a scalar.

1.3.4 Curl: Since two forms of vector multiplication exist, it is natural to expect another derivative form of **A**. The *curl* of **A** is defined as

$$\begin{aligned}\operatorname{curl}\mathbf{A} &= \nabla\times\mathbf{A}\\ &= \begin{vmatrix}\mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3\\ \dfrac{\partial(\ )}{\partial x_1} & \dfrac{\partial(\ )}{\partial x_2} & \dfrac{\partial(\ )}{\partial x_3}\\ A_1 & A_2 & A_3\end{vmatrix}\end{aligned}\qquad (1\text{-}12)$$

in determinant form.

1.4 Indicial Notation

One of the conveniences of modern treatments of the theory of elasticity is the use of shorthand notation to facilitate the mathematical manipulation of lengthy equations.

Referring to the ordered triple representation for **A** in Eq. (1-3), the three Cartesian components can be symbolized as A_i, where the subscript or *index* i is understood to take the sequential values 1, 2, 3. If we have nine quantities we may employ a double subscripted notation D_{ij}, where i and j range from 1 to 3 in turn. Later, we will associate these nine components with a higher form of a vector, called a tensor. Further, we may have 27 quantities, C_{ijk}, etc.

While i and j range as stated, an exception is made when two subscripts are identical, such as D_{jj}. The *Einstein summation convention* states that a subscript appearing *twice* is *summed* from 1 to 3. *No* subscript can appear more than *twice*. As an example, we have the inner product, Eq. (1-5), rewritten as

$$\begin{aligned}A_iB_i &= \sum_{i=1}^{3} A_iB_i\\ &= A_1B_1 + A_2B_2 + A_3B_3.\end{aligned}\qquad \text{(a)}\quad (1\text{-}13)$$

Also,

$$D_{jj} = D_{11} + D_{22} + D_{33}.\qquad \text{(b)}\quad (1\text{-}13)$$

It is apparent from the preceding examples that there are two distinct types of indices. The first type appears only once in each term of the equation and ranges from 1 to 3. It is called a *free* index. The second type appears twice in a single term and is summed from 1 to 3. Since it is immaterial which letter is used in this context, a repeated subscript is called a *dummy* index. That is, $D_{ii} = D_{jj} = D_{kk}$.

From the preceding discussion, it may be deduced that the number of individual terms represented by a single product is 3^k, where k is the number of free indices.

There are some situations in which double subscripts occur where the summation convention is not intended. This is indicated by the symbol $\not\Sigma$. For example, the individual components D_{11}, D_{22} and D_{33} could be represented by D_{ii}, $\not\Sigma_i$.

The *product* of the three components of a vector is expressed by the Pi convention:

$$\prod_{i=1}^{3} A_i = A_1 A_2 A_3. \tag{1-14}$$

Partial differentiation may also be abbreviated using the *comma* convention

$$\frac{\partial A_i}{\partial x_j} = A_{i,j}. \tag{1-15}$$

Since both i and j are free indices, Eq. (1-15) represents $3^2 = 9$ quantities. With repeated indices,

$$\begin{aligned} \frac{\partial A_i}{\partial x_i} &= A_{i,i} \\ &= \operatorname{div} \mathbf{A} \end{aligned} \tag{1-16}$$

as defined in Eq. (1-11).

Further,

$$\begin{aligned} \frac{\partial D_{ij}}{\partial x_j} &= D_{ij,j} \\ &= D_{i1,1} + D_{i2,2} + D_{i3,3}, \end{aligned} \tag{1-17}$$

which takes on $3^1 = 3$ values for each $i = 1, 2, 3$. This example combines the summation and comma conventions.

1.5 Coordinate Rotation

In Fig. 1-1 (see [1.2]), we show a position vector to point P, $\mathbf{r}$, resolved into components with respect to *two* Cartesian systems, x_i and x_i', having a common origin. The unit vectors in the x_i' system are shown as $\mathbf{e}_i'$ on the figure.

First, we consider the point P with coordinates $P(x_1, x_2, x_3) = P(x_i)$ in the unprimed system and $P(x_1', x_2', x_3') = P(x_i')$ in the primed system. The linear transformation between the coordinates of P is given by

$$\begin{aligned} x'_1 &= \alpha_{11}x_1 + \alpha_{12}x_2 + \alpha_{13}x_3 \\ x'_2 &= \alpha_{21}x_1 + \alpha_{22}x_2 + \alpha_{23}x_3 \\ x'_3 &= \alpha_{31}x_1 + \alpha_{32}x_2 + \alpha_{33}x_3 \end{aligned} \qquad \text{(a)} \quad \text{(1-18)}$$

or

$$x'_i = \alpha_{ij}x_j \qquad \text{(b)} \quad \text{(1-18)}$$

using the summation convention. Each of the nine quantities α_{ij} is the cosine of the angle between the ith primed and the jth unprimed axis, that is,

$$\alpha_{ij} = \cos(x'_i, x_j) = \frac{\partial x_j}{\partial x'_i} = \mathbf{e}'_i \cdot \mathbf{e}_j = \cos(\mathbf{e}'_i, \mathbf{e}_j) \qquad \text{(1-19)}$$

The α_{ij}'s are known as *direction cosines* and are conveniently arranged in tabular form for computation:

	x_1	x_2	x_3
x'_1	α_{11}	α_{12}	α_{13}
x'_2	α_{21}	α_{22}	α_{23}
x'_3	α_{31}	α_{32}	α_{33}.

(1-20)

It is emphasized that, in general, $\alpha_{ij} \neq \alpha_{ji}$. From a computational standpoint, Eq. (1-19) indicates that expressing the unit vectors in the x'_i coordinate system, $\mathbf{e}'_i$, in terms of those in the x_i system, $\mathbf{e}_i$, is tantamount to evaluating the corresponding α_{ij} terms. A numerical example is given in Sections 2.4.3 and 2.5.

We next consider the position vector $\mathbf{r}$ and recognize that the *components* are related by Eq. (1-18). Conversely, any quantity which obeys this transformation law is a vector. This somewhat indirect definition of a vector proves to be convenient for defining higher-order quantities, *Cartesian tensors.*

From a computational standpoint, it is often convenient to carry out the transformations indicated in Eq. (1-18) in matrix form as

$$\{\mathbf{x}'\} = [\mathbf{R}]\{\mathbf{x}\}, \qquad \text{(1-21)}$$

in which

$$\{\mathbf{x}'\} = \{x'_1\ x'_2\ x'_3\} \qquad \text{(a)}$$

$$\{\mathbf{x}\} = \{x_1\ x_2\ x_3\} \qquad \text{(b)} \quad \text{(1-22)}$$

$$[\mathbf{R}] = \begin{bmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{bmatrix}, \qquad \text{(c)}$$

$[\mathbf{R}]$ is called a *rotation* matrix.

1.6 Cartesian Tensors

A tensor of order n is a set of 3^n quantities which transform from one coordinate system, x_i, to another, x'_i, by a specified law, as follows:

n	order	transformation law
0	zero (scalar)	$A'_i = A_i$
1	one (vector)	$A'_i = \alpha_{ij} A_j$
2	two (dyadic)	$A'_{ij} = \alpha_{ik}\alpha_{jl} A_{kl}$
3	three	$A'_{ijk} = \alpha_{il}\alpha_{jm}\alpha_{kn} A_{lmn}$
4	four	$A'_{ijkl} = \alpha_{im}\alpha_{jn}\alpha_{kp}\alpha_{lq} A_{mnpq}$

Order zero and order one tensors are familiar physical quantities, whereas the higher-order tensors are useful to describe physical quantities with a corresponding number of associated directions.

Second-order tensors (dyadics) are particularly prevalent in elasticity and the transformation may be carried out in a matrix format, analogous to Eq. (1-21), as

$$[\mathbf{A}'] = [\mathbf{R}][\mathbf{A}][\mathbf{R}]^T, \tag{1-23}$$

in which

$$[\mathbf{A}'] = \begin{bmatrix} A'_{11} & A'_{12} & A'_{13} \\ A'_{21} & A'_{22} & A'_{23} \\ A'_{31} & A'_{32} & A'_{33} \end{bmatrix} \tag{1-24}$$

and [**A**] is similar.

It may be helpful to visualize a tensor of order n as having n unit vectors or directions associated with each component. Thus, a scalar has no directional association (isotropic) and a vector is directed in one direction. A second-order tensor has two associated directions, perhaps one direction *in* which it acts and another defining the surface *on* which it is acting.

1.7 Algebra of Cartesian Tensors

Tensor arithmetic and algebra are similar to matrix operations in regard to addition, subtraction, equality and scalar multiplication. Multiplication of two tensors of order n and m produces a new tensor of order $n + m$, for example,

$$A_i B_{jk} = C_{ijk} \tag{1-25}$$

For repeated indices the summation convention holds, as shown in Eq. (1-13b).