Volume 1BC

CALCULUS OF ONE VARIABLE

Kenneth McAloon/ Anthony Tromba

EAGLE MATHEMATICS SERIES

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CALCULUS OF ONE VARIABLE

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HARCOURT BRACE JOVANOVICH, INC.

New York *Chicago* *San Francisco* *Atlanta*

ISBN: 0-15-518525-X

Library of Congress Catalog Card Number: 72-186225

Printed in the United States of America

CALCULUS OF ONE VARIABLE

EAGLE MATHEMATICS SERIES

A series of textbooks for an undergraduate program in mathematical analysis

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VOLUMES

1AB *Calculus of Elementary Functions*

1BCD *Calculus*

1BC *Calculus of One Variable*

2A *Linear Algebra*

PREFACE

The introductory calculus course has come to assume an unusual number of burdens. The student must master a bewildering variety of problems, and the instructor is expected to communicate the spirit and cogency of modern mathematical rigor. This book, which is the product of a collaboration of teachers and undergraduates, is designed to help the instructor to teach the course coherently and to allow the student to develop strong intuitions. All the essential topics of the basic one-year course are covered. We have emphasized geometric understanding, because this is a form of rigorous thinking with which the student is already familiar. The result is a presentation of the calculus which is sometimes very classical and, other times, rather innovative.

The text begins with a discussion of functions and some elementary analytic geometry. A special section on "Unsolvable Problems of Analytic Geometry" anticipates the study of the calculus. We first introduce limits as a technique for solving geometric problems involving moving points and then lead the reader to a rigorous definition of limit in terms of open interval neighborhoods. The proofs of basic limit theorems are relatively simple when formulated in terms of neighborhoods, and the abundance of illustrations truly enables the student to "see" the proof. The geometric intuition about limits makes the transition to computing tangents and the derivative very natural and the student can appreciate the many applications of these ideas.

We assume that the reader knows high-school algebra, geometry, and the rudiments of trigonometry. For the sake of simplicity, we have kept set theory to a minimum; about the only sets used are intervals. Functions are defined as rules and are written using the elegant $f\colon \mapsto f(x)$

notation. Occasionally, we use the symbol $\Rightarrow$ to signify implication; we indicate the end of a proof by ∎. From the onset, in addition to the rational functions and various ad hoc examples, we use the trigonometric functions in our discussions (for reference, basic facts about these functions are displayed in §8.1). As a result, there is a much broader range of available examples which, in turn, reinforces the case for the precise formulation of the notions of function, limit, derivative, and so on.

The presentation of the definite integral also begins in an intuitive setting (Chapter 6). Working from a geometric definition based on the concept of area, we carefully set out the basic properties of the integral which are needed to prove the Fundamental Theorem of Calculus. This important result is established and then applied to area and volume problems. To facilitate working with integrals, we include an entire section on functions defined by integrals.

In Chapter 7 we are ready to turn to the task of formally defining the definite integral in terms of Riemann sums. There, when the integral is considered as a limit of sums, the basic properties needed to prove the Fundamental Theorem are established rigorously. Finally, we use the method of Riemann sums for those applications which expressly require it—arc length, surface area, and work.

Chapter 8 is a thorough study of the transcendental functions and Chapter 9 treats the various techniques of integration. If, for certain courses, it is desirable to teach these topics early, Chapters 8 and 9 can precede Chapter 7.

The final three chapters are, for all practical purposes, independent of one another. Chapter 10 covers infinite series; the final section, §10.6, deals with sequences and series of functions and is intended for more advanced students. In Chapter 11 we introduce the study of ordinary differential equations and rely extensively on physical examples to explain how such equations arise (the topic of oscillations is treated at some length). Although we use only elementary methods to handle the first-order linear equation and the second-order linear equation with constant coefficients, we do apply the method of power series to second-order linear equations with variable coefficients (thus, §11.6 requires Chapter 10).

Chapter 12, Numerical Methods, is of special interest to courses with students from the computer sciences. The topics we consider here include polynomial interpolation, numerical quadrature (the section on Gaussian quadrature is optional), finite difference interpolation, and differential equations. In addition, there is a formula for numerical differentiation based on Stirling's interpolation formula. This subject matter lends itself remarkably well to a course at this level, demonstrating the wide applicability of elementary calculus in a most effective way.

The exercises—perhaps the most important feature of a calculus book—are plentiful and varied. There is an exercise set at the end of each section and review exercises for each chapter. The exercises are tightly integrated to the text; when possible, we have included problems which illustrate the uses of calculus in different scientific areas. We denote more challenging problems with an asterisk. Many exercises contain instructive hints and selected answers are to be found at the back of the book. There are also tables of values of the transcendental functions and comprehensive listings of derivatives and integrals in the backmatter.

The authors would like to express their especial appreciation to the editor of this volume, Miss Marilyn Davis, whose searching comments and diligence were integral to their work, and also to Mr. Harry Rinehart for his design of the book.

Kenneth McAloon
Anthony Tromba

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1

FUNCTIONS

In the modern treatment of calculus, function and limit are central concepts. The idea of function is fundamental in mathematics, and in the applications of mathematics to the physical and social sciences. In this chapter we discuss functions and some additional concepts that will be used throughout the book. We present a brief exposition of analytic geometry—by no means an exhaustive one, but more than sufficient for our needs. In the closing section we introduce some problems that motivate the study in the rest of the book. Limits are considered in Chapter 2.

1 INTERVALS AND FUNCTIONS

Analytic geometry deals with techniques of describing geometric facts in the language of numbers. The simplest example of this is the way we associate a **real number** with each point on a straight line. We may draw a "real number line" as in Figure 1-1. Every point on the line represents a

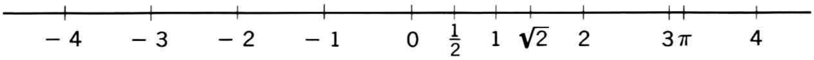

FIGURE 1-1

different real number, and each real number is represented by a different point on the line. We, therefore, can speak of real numbers as points. When we refer to the **real line**, it is this picture that we have in mind.

As the figure indicates, the **integers**, $0, 1, 2, \ldots, -1, -2, -3, \ldots$ correspond to an evenly spaced sequence of points on the real line. The other rational numbers, that is, numbers that can be expressed as fractions with integers in both the numerator and the denominator, are also points on the line. The rational numbers do not exhaust the points on the real line, however; numbers corresponding to the remaining points are called irrational numbers. Examples of irrational numbers are π, $\sqrt{2}$, and $\sqrt[3]{37}$.

In the above paragraph we defined certain collections of real numbers —the integers, the rational numbers, the irrational numbers. Such collections are called **sets**. In general, a set is any collection of objects and the members of a set are called its elements. For example the collection of real numbers itself is a set and we denote this set by $\boldsymbol{R}$. The number π is a real number and π is an element of $\boldsymbol{R}$. The set of natural numbers is the set whose elements are $1, 2, 3, \ldots$; we denote this set by $\boldsymbol{N}$. The set of natural numbers coincides with the set of positive integers.

To indicate that the element x is a member of the set X we write $x \in X$; when x is not an element of the set X we write $x \notin X$. Thus, for example, we have $6 \in \boldsymbol{N}$, since 6 is a natural number; we also have $6 \in \boldsymbol{R}$, since 6 is a real number. However, $-\sqrt{2}$ is not a natural number, so we have $-\sqrt{2} \notin \boldsymbol{N}$; it is still the case that $-\sqrt{2} \in \boldsymbol{R}$.

Given two real numbers a and b, with $a < b$ (read "a less than b"), we call that portion of the real line between a and b the open interval from a to b, and we write $]a, b[$. Neither a nor b is an element of $]a, b[$; only those numbers which lie between them are elements of this open interval. Algebraically, the open interval $]a, b[$ may be described as the set of all real numbers c which satisfy the two inequalities

$$a < c \quad \text{and} \quad c < b$$

The numbers a and b are called the endpoints of the open interval $]a, b[$.

If we "tack on" the endpoints of the open interval, that is, if we consider the set of all real numbers c satisfying the inequalities $a \leqslant c$ and $c \leqslant b$ (read "a less than or equal to c" and "c less than or equal to b"), then we call the resulting set of real numbers the closed interval from a to b, and write $[a, b]$. Again, the numbers a and b are called the endpoints of the interval. It is important to notice that a closed interval contains both of its endpoints, while an open interval contains neither.

We also define the half-open intervals $[a, b[$ and $]a, b]$. The first set is just $]a, b[$ with the extra point a included, and the second is $]a, b[$ with the number b included. A half-open interval contains exactly one of its two endpoints, either a or b.

We have assigned a meaning to the symbols $[a, b]$, $]a, b[$, $]a, b]$, and $[a, b[$ only when a and b are real numbers with $a < b$. We shall also find it convenient to define the following similar symbols:

$]-\infty, a[$ means "the set of all real numbers less than a";
$]-\infty, a]$ means "the set of all real numbers less than or equal to a";
$]a, \infty[$ means "the set of all real numbers greater than a";

$[a, \infty[$ means "the set of all real numbers greater than or equal to a"; and
$]-\infty, \infty[$ means "the set of all real numbers."

Geometrically, $]a, \infty[$ represents that portion of the real number line which lies to the right of a, not including a itself; similar geometric pictures can be drawn for the other "infinite intervals." We assign no meaning to the symbols $-\infty$ and ∞ when occurring alone, nor do we write symbols like $[-\infty, a[$ or $[a, \infty]$. The intervals $]-\infty, a[$, $]a, \infty[$, and $]-\infty, \infty[$ are all **open**; the intervals $]-\infty, a]$ and $[a, \infty[$ are **half open**. The number a is the **only** endpoint of the intervals $[a, \infty[$, $]a, \infty[$, $]-\infty, a]$, and $]-\infty, a[$. The interval $]-\infty, \infty[$ has no endpoints.

Example 1 The interval $]3, \pi[$ contains the real numbers 3.1, 314/1000, and $\pi - 0.001$, but does not contain the real numbers $3, \pi, \pi + 0.0001$, or 117. The interval $]0, \infty[$ contains all positive real numbers; the interval $[0, \infty[$ contains all nonnegative real numbers. The interval $]-7, 2]$ has endpoints -7 and 2, and it contains 2 but not -7.

Example 2 If J is the interval $[-1, 1]$, then J contains $-1, 0, 0.5$, and 1. If K is the interval $]-\infty, -2]$, then K contains any real number less than or equal to -2. For instance, -12 belongs to K.

Example 3 None of the following are meaningful symbols for intervals: $]-3, -5[$, $[6, 0]$, and $]3, \infty]$.

If I is any interval and c is a point in I, then c is an element of I, and we have $\boldsymbol{c \in I}$. If c is not a point in I, then we have $\boldsymbol{c \notin I}$.

Example 4 According to Example 1, $3.1 \in]3, \pi[$, but $3 \notin]3, \pi[$ and $\pi \notin]3, \pi[$.

Example 5 For any real number c, we have $c \in]0, \infty[$ if and only if $c > 0$; and we have $c \in]a, b[$ if and only if $a < c < b$.

Any open interval containing the real number c is called a **neighborhood** of c.

Example 6 The open interval $]0, 10[$ is a neighborhood of 5, 7, and $\sqrt{46}$, but not of 0, -35, or $\sqrt{100}$.

Example 7 The infinite interval $]-\infty, a[$ is a neighborhood of any real number less than a; $]-\infty, \infty[$ is a neighborhood of every real number.

Example 8 The interval $]3, 5]$ is not a neighborhood of 4, although $4 \in]3, 5]$, since $]3, 5]$ is not an open interval.

Example 9 If c is any real number, and if e is a **positive** real number, then $]c - e, c + e[$ is a neighborhood of c.