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Geometry of Numbers

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GEOMETRY OF NUMBERS

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To our teacher, colleague and
dear friend Edmund Hlawka

PREFACE

This book deals with bodies and lattices in the n-dimensional euclidean space. The bodies considered are convex bodies centered at the origin or, more generally, star bodies (with respect to the origin). With each star body there is associated a continuous distance function; it is a positively homogeneous function assuming the value 1 at the points of the boundary of the given body.

The correspondence between star bodies and distance function just sketched brings on the interchange of the geometric and the arithmetic viewpoint that is typical for the subject. Historically, the arithmetic viewpoint existed first. But the geometry of numbers as such came into being only when MINKOWSKI [GZ] brought in the geometric viewpoint. A number of later works, a.o. those by REMAK, OPPENHEIM, DAVENPORT and BARNES, are plainly of an arithmetic character or give proof of an arithmetizing tendency. On the other hand, the rôle and the fruitfulness of geometric ideas is apparent in the works of BLICHFELDT, MORDELL, MAHLER and ROGERS. In the present monograph the geometric viewpoint is prevalent. Circumstantial digressions of a computational nature have not been inserted. Furthermore, several proofs available in the literature have been remodeled on more geometric lines.

A basic problem in the geometry of numbers may be stated as follows. Under which conditions does a given convex body, or star body, contain a point with integral coordinates, not all zero? One may also ask under which conditions, for each point z of the space, there is a point x in a given body such that $x-z$ is a point with integral coordinates. These two problems give rise to the introduction of the arithmetical (or homogeneous) and the inhomogeneous minimum of a distance function with respect to the lattice of points with integral coordinates. Instead of the last lattice one may work with an arbitrary lattice as well. One can then define the absolute homogeneous minimum and (as they are called in this book) the lower and upper absolute inhomogeneous minimum of a distance function. Geometrically speaking, these quantities correspond with the critical determinant, the covering constant and the inhomogeneous determinant of a star body. The study of these quantities is the object of the geometry of numbers.

It is the aim of the present book to give a systematic account of our present knowledge in the field just indicated. It contains a detailed exposition of the general theory and presents complete proofs of all the main results. Further results known in the literature are commented throughout the book. Sometimes, for more details, the reader is referred to the existing books on the subject, in particular CASSELS' book and KELLER's encyclopaedia article. Some topics discussed at length are VORONOÏ polyhedra, polar reciprocal and compound convex bodies, anomaly of a star body, packing and covering, methods of BLICHFELDT and MORDELL, COXETER's reflexible forms, MARKOV's minimal forms, and asymmetric inequalities. There are two sections (secs. 45 and 51) on diophantine approximation. Considerable attention is given to inhomogeneous problems. Analogues of the geometry of numbers in (finite-dimensional) spaces over the field of complex numbers, non-archimedean fields, or the ring of adèles are not considered. The bibliography is fairly complete as to the period ranging from the year 1935 (the year of publication of KOKSMA's Ergebnisse report) to the year 1965.

About the arrangement of the material the following may be said. A long chapter (chapter 2) is devoted nearly exclusively to convex bodies and points with integral coordinates. There is a separate chapter on MAHLER's theory of star bodies. The last two chapters deal with arithmetical problems.

The prerequisites necessary for the understanding of the book are rather modest. They only comprise elementary real analysis and some basic facts about algebraic number fields (exposed briefly in section 4), measure theory and topological spaces. At a few places use is made of the theory of continued fractions.

The sections of the book are numbered consecutively; they are divided into subsections. The numbering of the definitions, theorems and formulas starts afresh in each new section; if they are referred to in a different section, then the number of that section is placed before the number of the definition, theorem or formula. Capital letters between square brackets refer to Part A of the bibliography (books and monographs). A number and a letter between square brackets refer to Part B (papers). Here, the convention is that the number is that of the (main) section where the paper in question is commented. Thus, the bibliography may equally well serve as an author index.

The author is indebted to the late Professor Koksma who stimulated him to write this book. Most of the bibliographical research was done when he was a scientific officer in the Department of Pure Mathematics of the Mathematical Centre at Amsterdam. His thanks are due to Mrs. Troelstra and Mrs. van Proosdij for preparing the typescript. Finally, he thanks the editors of the "Bibliotheca Mathematica" for taking up the book in their series.

Amsterdam, July 1969 GERRIT LEKKERKERKER

Preface to the second edition

The second edition of this book was prepared jointly by P. M. Gruber and the author of the first edition. The authors decided to retain the existing text, with minor corrections, and to add to each chapter supplementary sections on the more recent developments. While this has obvious drawbacks, it has the definite advantage of showing cleariy where recent progress took place and in what areas interesting results in the future may be expected.

The development of the geometry of numbers in the last two decades has some remarkable features. In the 60's progress was slow. The main problems were either solved or seemed untractable. The attempts, for instance, to improve upon the Minkowski–Hlawka theorem or to settle the famous conjecture on the product of non-homogeneous linear forms met with little success. In the 70's, however, the situation has changed drastically and the interest in the field began to rise again. The change was brought about by two factors. On the one hand significant progress was achieved in classical problems such as the ball packing problem, the conjecture on non-homogeneous linear forms, the determination of the homogeneous and non-homogeneous minima of quadratic forms and the analysis of the Markov spectrum. On the other hand new and vigorous branches of the geometry of numbers developed; we mention only lattice polytopes and zeta functions.

Progress was not uniform; this can be seen from the different lengths of the supplements to the various chapters. Much progress was achieved in geometric areas such as packing, covering and tiling which are close to discrete geometry and crystallography. Another feature, common in present day mathematics, is the appearance of many new and unexpected links with other branches of mathematics. Notable examples are modular forms, coding theory and numerical integration.

While a complete coverage of the geometry of numbers is out of reach we hope to have given a fearly good account of what is going on in this area. The book can thus be used as a reference work but also as an advanced introduction to geometry of numbers. It is hoped that the many implicitly of explicitly stated open problems will contribute to further research in this area.

We have tried to arrange the material such that the new sections can be read independently from the original text. A reader looking for an easy introduction to the subject is invited to first consult the joint booklet on solved and unsolved lattice point problems by one of the authors with Erdös and Hammer which is broader in scope and much more elementary.

Whereas in the 40's and 50's the centres of the geometry of numbers were Manchester, Cambridge, London and Vienna, in more recent times much progress was achieved in Columbus, Chandigarh, Adelaide and in Moscow and Leningrad.

We are very grateful to many colleagues and friends in these and other places who helped us in the prepration of this edition. In particular we should like to thank Professors Bambah, Chalk, Coxeter, Dumir, Groemer, Hans-Gill, Hlawka, Lenstra, Malyšev, McMullen, Rankin, Ryškov, Schnitzer, Seidel, Skubenko, White, Wills and Zassenhaus and Doctors Betke, G. Fejes Tóth, Horváth, Müller, Nowak, Ramharter, Sorger, Szabó and Temesvari.

Each of the two authors has a different mathematical background and different interests. Through many (often difficult, but always temperate and friendly) discussions we have tried to give the new sections a balanced appearance. We hope that still existing inhomogeneities rather improve than diminish the liveliness of the presentation.

The joint work was made possible through several visits of G. Lekkerkerker to Vienna and of P. Gruber to Amsterdam. We gratefully acknowledge the support of ZWO, the Dutch Organisation for the Advancement of Pure Science, which made possible the stays at Amsterdam.

Finally, our gratitudes goes to Susanna Nagy who prepared the typescript and to the publisher who met our wishes.

PETER M. GRUBER and C. GERRIT LEKKERKERKER

CONTENTS

CHAPTER 1. PRELIMINARIES

SUPPLEMENT TO CHAPTER 1

CHAPTER 2. CONVEX BODIES AND LATTICE POINTS

SUPPLEMENT TO CHAPTER 2

CHAPTER 3. THE CRITICAL DETERMINANT, THE COVERING CONSTANT AND THE INHOMOGENEOUS DETERMINANT OF A SET

SUPPLEMENT TO CHAPTER 3

CHAPTER 4. STAR BODIES

SUPPLEMENT TO CHAPTER 4

CHAPTER 5. SOME METHODS

NOTE ON CHAPTER 5

CHAPTER 6. HOMOGENEOUS FORMS

SUPPLEMENT TO CHAPTER 6

CHAPTER 7. INHOMOGENEOUS FORMS

SUPPLEMENT TO CHAPTER 7

BIBLIOGRAPHY

CHAPTER 1

PRELIMINARIES

The geometry of numbers to which this book is devoted deals with arbitrary bodies and arbitrary lattices in the n-dimensional euclidean space. Its aim is to study various quantities describing the behaviour of a body with respect to a lattice.

In this chapter we expose a number of basic properties of convex bodies, star bodies and lattices. The last section will be devoted to algebraic number fields.

1. Notations. Convex bodies

1.1. Throughout this book we use the following notations concerning points and sets.

The n-dimensional euclidean space is denoted by R^n. Its points are sometimes considered as vectors. Its dimension n will nearly always be $\geqq 2$.

Points of R^n are always denoted by small Latin letters; we use the letters u, v, w, x, y, z, a, b, c, d, e, and employ accents or upper indices. In particular, the letters u, v, w are used for points with integral coordinates. The coordinates of a vector are denoted by the same letter, with a lower index $1, \ldots, n$; if a letter denoting a vector has already an (upper) index, then that index is put after the coordinate index. Thus:

$$a = (a_1, \ldots, a_n), \quad a' = (a'_1, \ldots, a'_n), \quad x^k = (a_{1k}, \ldots, x_{nk}).$$

If powers of coordinates are considered, then brackets are used: $(x_1)^2$, $(x_i)^4$. Except for coordinates of points in R^n and elements of matrices, real numbers are denoted by small Greek letters. Integers are denoted by $i, j, k, l, m, n, p, q, r, s, t$, with or without lower indices. However, i may stand for $\sqrt{-1}$ and s_{ij}, t_{ij} may be elements of (real symmetric) matrices (s_{ij}), (t_{ij}). If α is real, then $[\alpha]$ denotes the largest integer $\leqq \alpha$. The letters f, g, h are used to denote so-called distance functions of convex bodies or star bodies; otherwise, functions are indicated by small Greek letters and, in a number of cases, by capitals.

The origin is denoted by o. For $i = 1, \ldots, n$, the point which has ith coordinate 1, whereas the other coordinates are all zero, is denoted by e^i. Further, vectorial sums and differences $x \pm y$ and multiples αx are used. The length of a vector x is

$$|x| = \{(x_1)^2 + \cdots + (x_n)^2\}^{\frac{1}{2}},$$

and the inner product of x and y is

$$x \cdot y = x_1 y_1 + \cdots + x_n y_n.$$

Next, $k \leqq n$ points $x^1, \ldots, x^k$ are called *independent* if they do not belong to a linear subspace of R^n of dimension less than k; the linear subspace generated by k independent points $x^1, \ldots, x^k$ is denoted by $L(x^1, \ldots, x^k)$. For $k = 0$, by the last expression we mean the set consisting of the single point o.

Arbitrary sets in R^n are denoted by capital letters. If a set M is measurable (in the sense of Lebesgue), then its Lebesgue measure is called the *volume* of M and denoted by $V(M)$ or V. The set consisting of finitely many points $x^1, \ldots, x^k$ is denoted by $\{x^1, \ldots, x^k\}$; in particular, $\{x\}$ is the set consisting of the single point x. The set of points x for which some given property $P(x)$ holds, is denoted by $\{x: P(x)\}$. The set of values of any function φ on this set is written as $\{\varphi(x): P(x)\}$; a similar notation is used for functions of other variables than points in R^n. Furthermore, the following notations are used:

$M + x$: the set of points $y + x$ with $y \in M$,
αM: the set of points αx, $x \in M$ (α real),
$M_1 + M_2$: the set of points $x + y$ with $x \in M_1$, $y \in M_2$.

It should be observed that, in general, the sets $M + M$ and $2M$ are not identical. The set $M - M$ is also denoted by $\mathscr{D}M$ and is called the *difference set* of M.

The symbols $\cap$, $\cup$, $\subset$ are used to denote the set-theoretical intersection and union and the inclusion relation, respectively. Further, $\emptyset$ denotes the empty set, and $M_1 \setminus M_2$ denotes the set of points which belong to M_1 but not to M_2.

An open connected set M in R^n is called a *region*. If a set M has inner points and is contained in the closure of its open kernel, it is called a *body* or a *domain*. The open kernel of a set M is denoted by int M and the closure of M by $\overline{M}$.

1.2. A set H in R^n is called *convex*, if, for any two points $x, y \in H$, it contains all points of the segment joining x and y. If this property holds, then, in particular, $\frac{1}{2}(x+y) \in H$ if $x, y \in H$. Conversely, each point $x \in H$ can be written as $\frac{1}{2}(x+x)$. So we have the formula

$$(1) \qquad H+H = 2H.$$

More generally, we have

$$(2) \qquad \alpha H+\beta H = (\alpha+\beta)H \qquad \text{if } \alpha, \beta > 0.$$

If a convex set is not contained in a hyperplane, i.e., if H contains $n+1$ independent points, then it has inner points. It is a body, because all inner points of any segment joining a point of $\overline{H}$ and a point of int H belong to int H.

A convex body H is called *strictly convex* if, for each two points $x, y \in \overline{H}$ $(x \neq y)$, all points $\vartheta x+(1-\vartheta)y$ with $0 < \vartheta < 1$ are inner points of H.

Examples of convex bodies in R^n are (solid) spheres, cubes and, more generally, ellipsoids and parallelotopes*. A convex polyhedron P is the intersection of finitely many half-spaces, e.g., the half-spaces which contain the polyhedron P and whose bounding hyperplanes contain the $(n-1)$-dimensional faces of P. Conversely, the intersection of arbitrarily many half-spaces is always a convex set; if the number of half-spaces is finite, it is a convex polyhedron. Further examples are convex cones (cylinders) possibly truncated by a hyperplane (two parallel hyperplanes).

The orthogonal projection of a convex H onto a hyperplane is again convex. If H is closed and H' is the projection of H onto the plane $x_n = 0$, then H is given by two inequalities of the type

$$(3) \qquad \varphi_1(x_1, \ldots, x_{n-1}) \leqq x_n \leqq \varphi_2(x_1, \ldots, x_{n-1}),$$

where $(x_1, \ldots, x_{n-1})$ runs through H' and φ_1, φ_2 are certain real functions defined on H'. Clearly, the function φ_1 is convex and the function φ_2 is concave.

We shall often have to consider so-called tac-planes of a given convex body. In general, a hyperplane P is called a *tac-plane* to a set M if P contains at least one point of $\overline{M}$ and M is contained in one of the (closed) half-spaces bounded by P. A *tac-strip* is a (closed) strip bounded by two

* Instead of 'parallelotope' we shall often use the word 'cell'.