SECOND EDITION

SPECIAL FUNCTIONS OF MATHEMATICS FOR ENGINEERS

LARRY C. ANDREWS

Special Functions of Mathematics for Engineers

Larry C. Andrews
University of Central Florida

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Preface to the Second Edition

The primary changes in this second edition include the introduction of many more applications, chosen from a variety of fields such as statics, dynamics, statistical communication theory, fiber optics, heat conduction in solids, vibration phenomena, and fluid mechanics, among others. In many cases these applications appear in the chapter in which the particular special function is introduced. However, because applications involving Bessel functions and hypergeometric-type functions are far more extensive than those of the other functions, they carry over to separate chapters devoted entirely to applications (Chaps. 8 and 12).

As in the first edition, the text is suitable for use either as a classroom text in various courses dealing with higher mathematical functions or as a reference text for practicing engineers and scientists. To this end I have tried to preserve the readability of the first edition, improving it where I could by the addition of further examples or clearer exposition. For instance, I have rearranged the order of topics in Chap. 1 so that asymptotic formulas follow the discussion of improper integrals, and in addition to the chapter on applications, the discussion of Bessel functions has been expanded to two chapters—one chapter devoted entirely to Bessel functions of the first and second kinds (Chap. 6) and one devoted to Bessel functions of other kinds, such as modified Bessel functions and spherical Bessel functions (Chap. 7). These discussions on Bessel functions also include some new material such as the introduction of addition formulas, Kelvin's functions, and Struve functions.

I am grateful to a number of students and colleagues for their helpful suggestions concerning this second edition. In particular, I wish to thank B. K. Shivamoggi, K. Vajravelu, and M. Belkerdid for their input concerning the choice of certain applications. I am further indebted to B. K. Shivamoggi for reading most of the new material

and offering many useful suggestions. Finally, I wish to thank the entire production staff of McGraw-Hill and, in particular, acknowledge my editor, Robert Hauserman, for his continued support of this project.

L. C. Andrews

Preface to the First Edition

Modern engineering and physics applications demand a more thorough knowledge of applied mathematics than ever before. In particular, it is important to have a good understanding of the basic properties of *special functions*. These functions commonly arise in such areas of application as heat conduction, communication systems, electro-optics, nonlinear wave propagation, electromagnetic theory, quantum mechanics, approximation theory, probability theory, and electric circuit theory, among others. Special functions are sometimes discussed in certain engineering and physics courses, and mathematics courses such as partial differential equations, but the treatment of special functions in such courses is usually too brief to focus on many of the important aspects, such as the interconnecting relations between various special functions and elementary functions. This book is an attempt to present, at the elementary level, a more comprehensive treatment of special functions than can ordinarily be done within the context of another course. It provides a systematic introduction to most of the important special functions that commonly arise in practice and explores many of their salient properties. I have tried to present the special functions in a broader sense than is often done by not introducing them as simply solutions of certain differential equations. Many special functions are introduced by the generating-function method, and the governing differential equation is then obtained as one of the important properties associated with the particular function.

In addition to discussing special functions, I have injected throughout the text by way of examples and exercises some of the techniques of applied analysis that are useful in the evaluation of nonelementary integrals, summing series, and so on. All too often in practice a problem is labeled "intractable" simply because the practitioner has not been exposed to the "bag of tricks" that helps the applied analyst deal with formidable-looking mathematical expressions.

During the last 10 years or so at the University of Central Florida we have offered an introductory course in special functions to a mix of advanced undergraduates and first-year graduate students in mathematics, engineering, and physics. A set of lecture notes developed for that course has finally led to this textbook. The prerequisites for our course are the basic calculus sequence and a first course in differential equations. Although complex variable theory is often utilized in studying special functions, knowledge of complex variables beyond some simple algebra and Euler's formulas is not required here. By not developing special functions in the language of complex variables, the text should be accessible to a wider audience. Naturally, some of the beauty of the subject is lost by this omission.

The text is not intended to be an exhaustive treatment of special functions. It concentrates heavily on a few functions, using them as illustrative examples, rather than attempting to give equal treatment to all. For instance, an entire chapter is devoted to the Legendre polynomials (and related functions), while the other orthogonal polynomial sets, including Hermite, Laguerre, Chebyshev, Gegenbauer, and Jacobi polynomials, are all lumped together in a single separate chapter. However, once the student is familiar with Legendre polynomials (which are perhaps the simplest set) and their properties, it is easy to extend these properties to other polynomial sets. Some applications occur throughout the text, often in the exercises, and Chap. 7 is devoted entirely to applications involving boundary-value problems. Other interesting applications which lead to special functions have been omitted, since they generally presuppose knowledge beyond the stated prerequisites.

Because of the close association of infinite series and improper integrals with the special functions, a brief review of these important topics is presented in the first chapter. In addition to reviewing some familiar concepts from calculus, this first chapter contains material that is probably new to the student, such as the Cauchy product, index manipulation, asymptotic series, Fourier trigonometric series, and infinite products. Of course, our discussion of such topics is necessarily brief.

I owe a debt of gratitude to the many students who took my course on special functions over the years while this manuscript was being developed. Their patience, understanding, and helpful suggestions are greatly appreciated. I want to thank my colleague and friend, Patrick J. O'Hara, who graciously agreed on several occasions to teach from the lecture notes in their early rough form, and who made several helpful suggestions for improving the final version of the manuscript. Finally, I wish to express my appreciation to Ken Werner, Senior Editor of Scientific and Technical Books Department, for his continued faith in this project and efforts in getting it published.

Notation for Special Functions

Notation	*Name of function*
Ai (x), Bi (x)	Airy functions of the first and second kinds
bei (x), ber (x), bei_p (x), ber_p (x)	Kelvin's functions
$B(x, y)$	Beta function
$B_x(p, q)$	Incomplete beta function
$b_n(x)$	Bessel polynomial
$C(x)$, $C_1(x)$, $C_2(x)$	Fresnel cosine integrals
$C_n^\lambda(x)$	Gegenbauer polynomial
Ci (x)	Cosine integral
cn u, dn u	Jacobian elliptic functions
$D_n(x)$	Parabolic cylinder function
Ei (x), $E_1(x)$	Exponential integral
$E(m)$	Complete elliptic integral of the second kind
$E(m, \phi)$	Elliptic integral of the second kind
$E(a_p; c_q; x)$	MacRobert E function
$\mathbf{E}_p(x)$	Weber function
erf x, erfc x	Error functions
$\zeta(x)$	Riemann zeta function
$F(a, b, c; x) = {}_2F_1(a, b; c; x)$	Hypergeometric function
$F(m, \phi)$	Elliptic integral of the first kind
${}_pF_q(a_p; c_q; x)$	Generalized hypergeometric function
$\Gamma(x)$	Gamma function
$\gamma(a, x)$, $\Gamma(a, x)$	Incomplete gamma functions
$G(a, b; c; x)$	Hypergeometric function of the second kind
$G_{p,q}^{m,n}\left(x \,\middle\vert\, {a_p \atop c_q}\right)$	Meijer G function

Notation	*Name of function*
$H_n(x), H_\nu(x)$	Hermite polynomial, Hermite function
$\mathbf{H}_p(x)$	Struve function of the first kind
$H_p^{(1)}(x), H_p^{(2)}(x)$	Hankel functions of the first and second kinds
$h_n^{(1)}(x), h_n^{(2)}(x)$	Spherical Hankel functions of the first and second kinds
$i_n(x)$	Modified spherical Bessel function of the first kind
$I_p(x)$	Modified Bessel function of the first kind
$\mathrm{Ji}_p\,(x)$	Integral Bessel function
$j_n(x)$	Spherical Bessel function of the first kind
$J_p(x)$	Bessel function of the first kind
$\mathbf{J}_p(x)$	Anger function
$\mathrm{kei}\,(x), \mathrm{ker}\,(x)$	Kelvin's functions
$K(m)$	Complete elliptic integral of the first kind
$k_n(x)$	Modified spherical Bessel function of the second kind
$K_p(x)$	Modified Bessel function of the second kind
$\mathrm{li}\,(x)$	Logarithmic integral
$L_n(x)$	Laguerre polynomial
$L_n^{(m)}(x), L_\nu^{(a)}(x)$	Associated Laguerre polynomial, associated Laguerre function
$\mathbf{L}_p(x)$	Modified Struve function
$M(a;c;x) = {}_1F_1(a;c;x)$	Confluent hypergeometric function
$M_{k,m}(x)$	Whittaker function of the first kind
$P_n(x), P_\nu(x)$	Legendre polynomial, Legendre function
$P_n^m(x)$	Associated Legendre function of the first kind
$P_n^{(a,b)}(x)$	Jacobi polynomial
$\Pi(m, a)$	Complete elliptic integral of the third kind
$\Pi(m, \phi, a)$	Elliptic integral of the third kind
$\psi(x)$	Digamma or psi function
$\psi^{(m)}(x)$	Polygamma function
$Q_n(x)$	Legendre function of the second kind
$Q_n^m(x)$	Associated Legendre function of the second kind
$\mathrm{Si}\,(x), \mathrm{si}\,(x)$	Sine integrals
$S(x), S_1(x), S_2(x)$	Fresnel sine integrals

Notation	*Name of function*
$\mathrm{sn}\, u$	Jacobian elliptic function
$T_n(x)$	Chebyshev polynomial of the first kind
$U_n(x)$	Chebyshev polynomial of the second kind
$U(a;c;x)$	Confluent hypergeometric function of the second kind
$W_{k,m}(x)$	Whittaker function of the second kind
$y_n(x)$	Spherical Bessel function of the second kind
$Y_p(x)$	Bessel function of the second kind

Special Functions of Mathematics for Engineers

ABOUT THE AUTHOR

Larry C. Andrews is a professor of mathematics at the University of Central Florida where he is also a member of the Department of Electrical Engineering. Dr. Andrews received a doctoral degree in theoretical mechanics in 1970 from Michigan State University. His current research interests include the propagation of laser beams through turbulent media, detection theory, signal processing, special functions, integral transforms, and differential equations.

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