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Shintani Zeta Functions

Akihiko Yukie

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To my parents Kenzo and Fumiko Yukie

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Preface

The content of this book is taken from my manuscripts 'On the global theory of Shintani zeta functions I–V' which were originally intended for publication in ordinary journals. However, because of its length and the lack of a book on prehomogeneous vector spaces, it has been suggested to publish them together in book form.

It has been more than 25 years since the theory of prehomogeneous vector spaces began. Much work has been done on both the global theory and the local theory of zeta functions. However, we concentrate on the global theory in this book. I feel that another book should be written on the local theory of zeta functions in the future.

The purpose of this book is to introduce an approach based on geometric invariant theory to the global theory of zeta functions for prehomogeneous vector spaces.

This book consists of four parts. In Part I, we introduce a general formulation based on geometric invariant theory to the global theory of zeta functions for prehomogeneous vector spaces. In Part II, we apply the methods in Part I and determine the principal part of the zeta function for Siegel's case, i.e. the space of quadratic forms. In Part III, we handle relatively easy cases which are required to handle the case in Part IV. In Part IV, we use the results in Parts I–III to determine the principal part of the zeta function for the space of pairs of ternary quadratic forms.

We expanded the introduction of the original manuscripts to help non-experts to have a general idea of the subject. What we try to discuss in the introduction is the history of the subject, and what is required to prove the existence of densities of arithmetic objects we are looking for. Even though the theory of prehomogeneous vector spaces involves many topics, we concentrate on two aspects of the theory, i.e. the global theory and the local theory, in the introduction.

Parts I–III of this book correspond to Parts I–III of the above manuscripts, and Part IV of this book corresponds to Parts IV and V of the above manuscripts. Since the manuscripts were originally intended for publication in ordinary journals, certain changes were made to make this book more comprehensible and self-contained.

However, it is impossible to make this book completely self-contained, and we have to require a reasonable background in adelic language, basic group theory, and geometric invariant theory. For this, we assume that the reader is familiar with the following four books and two papers

[1] A. Borel, Some finiteness properties of adele groups over number fields,
[2] A. Borel, *Linear algebraic groups*,
[28] G. Kempf, Instability in invariant theory,
[35] F. Kirwan, *Cohomology of quotients in symplectic and algebraic geometry*,
[46] D. Mumford and J. Fogarty, *Geometric invariant theory*,
[79] A. Weil, *Basic number theory*.

Weil's book [79] is a standard place to learn basic materials on adelic language. Since we do not depend on class field theory, it is enough for the reader to be familiar with the first several chapters of Weil's book. Borel's paper [1] is a place to learn properties of Siegel domains. We need two facts in geometric invariant theory. One is the Hilbert–Mumford criterion of stability, and the other is the rationality of the equivariant Morse stratification. Mumford–Fogarty [46] and Kirwan [35] are the standard books to learn geometric invariant theory and equivariant Morse theory. The rationality of the equivariant Morse stratification was proved by G. Kempf in his paper [28]. However, even though the proofs of the above two facts are technically involved, the statements of these facts are fairly comprehensible and do not require a special background to understand. Therefore, if the reader is unfamiliar with these subjects, I recommend the reader not to worry about the proofs of the statements in this book which we quote from geometric invariant theory and look at the above documents later if necessary.

We have three original results in this book. One is a generalization of 'Shintani's lemma' to GL(n) concerning estimates of the smoothed Eisenstein series. Shintani proved this lemma for GL(2) in [64]. The statement of the result is Theorem (3.4.31). The second result is the determination of the principal part of the zeta function for the space of quadratic forms. The statement of the result is Theorem (4.0.1). Shintani himself studied this case and determined the poles of the associated Dirichlet series for quadratic forms which are positive definite in [65]. The last and the main result of this book is the determination of the principal part of the zeta function for the space of pairs of ternary quadratic forms. The statement of the result is Theorem (13.2.2). We discuss the relevance of these results in the introduction.

D. Wright contributed to this book in many places. He suggested the use of 'Wright's principle' in §3.7 after he read the first manuscript of my paper [86]. Also §0.5 is largely from his note. He also found the reference concerning Omar Khayyam when we wrote our paper [84], and helped me to find some references in this book. I would like to give a hearty thanks to him. As I mentioned above, this book is based on geometric invariant theory. For this, I owe a great deal to D. Mumford for teaching me geometric invariant theory and equivariant Morse theory. I was staying at Institute for Advanced Study during the academic year 1989–1990, and at Sonderforschungsbereich 170 Göttingen during the academic year 1990–1991 while I was writing the manuscript of this book. I would like to thank them for their support of this project. This work was partially supported by NSF Grants DMS-8803085, DMS-9101091.

Akihiko Yukie
February 1992, Stillwater, Oklahoma, USA

Notation

For a finite set A, the cardinality of A is denoted by $\#A$. If f, g are functions on a set X and $|f(x)| \leq Cg(x)$ for some constant C independent of $x \in X$, we denote $f(x) \ll g(x)$. If $x, y \in \mathbb{R}$, we also use the classical notation $x \ll y$ if y is a much larger number than x. Since we use this classical notation only for numbers, and not for functions, we hope the meaning of this notation will be clear from the context.

Suppose that G is a locally compact group and Γ is a discrete subgroup of G contained in the maximal unimodular subgroup of G. For any left invariant measure dg on G, we choose a left invariant measure dg (we use the same notation, but the meaning will be clear from the context) on $X = G/\Gamma$ so that

$$\int_G f(g)dg = \int_X \sum_{\gamma \in \Gamma} f(g\gamma)dg.$$

We denote the fields of rational, real, and complex numbers by $\mathbb{Q}, \mathbb{R}, \mathbb{C}$ respectively. We denote the ring of rational integers by $\mathbb{Z}$. The set of positive real numbers is denoted by $\mathbb{R}_+$. For any ring R, $R^\times$ is the set of invertible elements of R. Let k be a number field, and o_k its integer ring. Let $\mathfrak{M}, \mathfrak{M}_\infty, \mathfrak{M}_\mathbb{R}, \mathfrak{M}_\mathbb{C}, \mathfrak{M}_f$ be the set of all the places, all the infinite, real, imaginary, finite places of k respectively. Let $\mathbb{A}_f$ (resp. $\mathbb{A}_f^\times$) be the restricted product of the k_v's (resp. $k_v^\times$'s) over $v \in \mathfrak{M}_f$. Let k_∞ (resp. $k_\infty^\times$) be the product of the k_v's (resp. $k_v^\times$'s) over $v \in \mathfrak{M}_\infty$. Then $\mathbb{A} = k_\infty \times \mathbb{A}_f$, $\mathbb{A}^\times = k_\infty^\times \times \mathbb{A}_f^\times$. If $x \in \mathbb{A}$ or $\mathbb{A}^\times$, we denote the finite (resp. infinite) part of x by x_f (resp. x_∞). If V is a vector space over k, we define $V_\mathbb{A}, V_\infty, V_f$ similarly. Let $\mathscr{S}(V_\mathbb{A}), \mathscr{S}(V_\infty), \mathscr{S}(V_f)$ be the spaces of Schwartz–Bruhat functions.

For any place v, k_v is the completion at v. If $v \in \mathfrak{M}_f$, $o_v \subset k_v$ is, by definition, the integer ring of k_v. Let $|\ |$ be the adelic absolute value. The absolute value of k_v is denoted by $|\ |_v$. For $x \in \mathbb{A}^\times$, we denote the product of the $|x|_v$'s over all $v \in \mathfrak{M}_f$ (resp. $v \in \mathfrak{M}_\infty$) by $|x|_f$ (resp. $|x|_\infty$). For $v \in \mathfrak{M}_f$, let π_v be the prime element, and $|\pi_v|_v = q_v^{-1}$. Note that if v is imaginary and $|x|$ is the usual absolute value, $|x|_v = |x|^2$.

Let r_1, r_2 be the numbers of real and imaginary places respectively. Let h, R, and e be the class number, regulator, and the number of roots of unity of k respectively. Let Δ_k be the discriminant of k. Let $\mathfrak{C}_k = 2^{r_1}(2\pi)^{r_2} hRe^{-1}$. We choose a Haar measure dx on $\mathbb{A}$ so that $\int_{\mathbb{A}/k} dx = 1$. For any finite place v, we choose a Haar measure dx_v on k_v so that $\int_{o_v} dx_v = 1$. We use the ordinary Lebesgue measure dx_v for v real, and $dx_v \wedge d\bar{x}_v$ for v imaginary. Then $dx = |\Delta_k|^{-\frac{1}{2}} \prod_v dx_v$ (see [79, p. 91]).

For $\lambda \in \mathbb{R}_+$, let $\underline{\lambda}$ be the idele whose component at v is $\lambda^{\frac{1}{[k:\mathbb{Q}]}}$ if $v \in \mathfrak{M}_\infty$ and 1 if $v \in \mathfrak{M}_f$. Clearly, $|\underline{\lambda}| = \lambda$. We identify $\mathbb{R}_+$ with a subgroup of $\mathbb{A}^\times$ by the map $\lambda \to \underline{\lambda}$. Let $\mathbb{A}^1 = \{x \in \mathbb{A}^\times \mid |x| = 1\}$. Then $\mathbb{A}^\times \cong \mathbb{A}^1/k^\times \times \mathbb{R}_+$, and $\mathbb{A}^1/k^\times$ is compact. We choose a Haar measure $d^\times t^1$ on $\mathbb{A}^1$ so that $\int_{\mathbb{A}^1/k^\times} d^\times t^1 = 1$. Using this measure, we choose a Haar measure $d^\times t$ on $\mathbb{A}^\times$ so that

$$\int_{\mathbb{A}^\times} f(t)d^\times t = \int_0^\infty \int_{\mathbb{A}^1} f(\underline{\lambda} t^1) d^\times \lambda d^\times t^1,$$

where $d^\times \lambda = \lambda^{-1} d\lambda$. For any finite place v, we choose a Haar measure $d^\times t_v$ on $k_v^\times$ so that $\int_{o_v^\times} d^\times t_v = 1$. Let $d^\times t_v(x) = |x|_v^{-1} dx$ if v is real, and $d^\times t_v(x) = |x|_v^{-1} dx \wedge d\bar{x}$ if v is imaginary. Then $d^\times t = \mathfrak{C}_k^{-1} \prod_v d^\times t_v$ (see [79, p. 95]).

Let $<>= \prod_v <>_v$ be a character of $\mathbb{A}/k$. Let v be a finite place. Suppose that $<>_v$ is trivial on $\pi_v^{-c_v} o_v$ and non-trivial on $\pi_v^{-c_v - 1} o_v$. Then we define $a_v = \pi_v^{c_v}$. Let $\mathrm{e}(x) = e^{2\pi\sqrt{-1}x}$. If v is a real place, there exists $a_v \in k_v^\times$ such that $< x >_v = \mathrm{e}(a_v x)$, and if v is an imaginary place, there exists $a_v \in k_v^\times$ such that $< x >_v = \mathrm{e}(a_v x + \overline{a_v x})$. For almost all v, $c_v = 0$. Let $\mathfrak{a} = (a_v)_v \in \mathbb{A}^\times$. Then $|\mathfrak{a}| = |\Delta_k|^{-1}$ (see [79, p. 113]). The idele $\mathfrak{a}$ is called the difference idele of k.

Let $\zeta_k(s)$ be the Dedekind zeta function. As in [79], we define

$$Z_k(s) = |\Delta_k|^{\frac{s}{2}} \left(\pi^{-\frac{s}{2}} \Gamma(\frac{s}{2})\right)^{r_1} \left((2\pi)^{-s} \Gamma(s)\right)^{r_2} \zeta_k(s).$$

We define $\mathfrak{R}_k = \mathrm{Res}_{s=1} Z_k(s)$.

For a character ω of $\mathbb{A}^\times / k^\times$, we define $\delta(\omega) = 1$ if ω is trivial, and $\delta(\omega) = 0$ otherwise.

Introduction

§0.1 What is a prehomogeneous vector space?

One contribution of Gauss to number theory in the early nineteenth century was the discovery of the correspondence between equivalence classes of integral binary quadratic forms and ideal classes of quadratic fields. This correspondence can be described as follows.

Let $f(v) = f(v_1, v_2) = x_0v_1^2 + x_1v_1v_2 + x_2v_2^2$ be a binary quadratic form such that x_0, x_1, x_2 are rational integers. We define an action of the group $\{\pm 1\} \times \mathrm{GL}(2, \mathbb{Z})$ on the set of integral binary quadratic forms so that if $g = (t, g_1)$ where $t = \pm 1, g_1 \in \mathrm{GL}(2, \mathbb{Z})$, $gf(v) = tf(vg_1)$. We consider equivalence classes of integral binary quadratic forms with respect to this action. It is easy to see that the discriminant $x_1^2 - 4x_0x_2$ is invariant under such an action. On the other hand, let m be a square free integer, and consider a non-zero ideal $\mathfrak{a}$ of the ring of algebraic integers in the field $k = \mathbb{Q}(\sqrt{m})$. The discriminant Δ_k of k is m if $m \equiv 1 \bmod 4$ and $4m$ if $m \equiv 2$ or $3 \bmod 4$. As a module over $\mathbb{Z}$, $\mathfrak{a}$ is generated by two elements, say α, β, because $\mathfrak{a}$ is a torsion free rank two module over $\mathbb{Z}$. Consider the binary quadratic form $f_{\mathfrak{a}}(v) = N(\mathfrak{a})^{-1}N(\alpha v_1 + \beta v_2)$, where $N(\mathfrak{a}), N(\alpha v_1 + \beta v_2)$ are the norms. It is easy to see that $f_{\mathfrak{a}}$ depends only on the ideal class of $\mathfrak{a}$. Moreover, it turns out that ideal classes of k correspond bijectively to equivalence classes of primitive integral binary quadratic forms with discriminant Δ_k by the map $\mathfrak{a} \to f_{\mathfrak{a}}$.

Gauss established this correspondence in [16], and the reader can see a modern proof in Theorem 4 [3, p. 142]. Here, we consider a natural question: why do we consider such a correspondence? One conceptual reason is that it gives us a parametrization of ideal classes of quadratic fields in terms of a group action on a vector space. We can use this parametrization to actually compute the class numbers of quadratic fields. But what we are interested in in this book is a more analytic question. In order to illustrate our purpose, let us describe the conjecture of Gauss.

Let h_d be the number of $\mathrm{SL}(2, \mathbb{Z})$-equivalence classes of primitive integral binary quadratic forms which are either positive definite or indefinite. Then Gauss conjectured the asymptotic property of the average of h_d. However, an integral form in the sense of Gauss is a form $x_0v_1^2 + 2x_1v_1v_2 + x_2v_2^2$ such that x_0, x_1, x_2 are integers. Here we consider $x_0v_1^2 + x_1v_1v_2 + x_2v_2^2$ such that x_0, x_1, x_2 are integers. With this understanding, we have the following asymptotic formula

(0.1.1)
$$\sum_{0<-d<x} h_d \sim \frac{\pi}{18\zeta(3)} x^{\frac{3}{2}},$$
$$\sum_{0<d<x} h_d \log \epsilon_d \sim \frac{\pi^2}{18\zeta(3)} x^{\frac{3}{2}}.$$

where $\epsilon_d = \frac{1}{2}(t + u\sqrt{d})$ and (t, u) is the smallest positive integral solution of the equation $t^2 - du^2 = 1$.

This conjecture was first proved by Lipschitz [42] for $d < 0$, and by Siegel [69] for $d > 0$, and much work has been done on the error term estimate also (see [65, pp. 44,45] for example). However, we are allowing all integers d here, and if $d = m^2d'$ and d' is a square free integer, $h_d, h_{d'}$ are related by a simple relation. Therefore, we are counting essentially the same object infinitely many times in (0.1.1). If k is a quadratic field over $\mathbb{Q}$, let h_k, R_k be the class number and the regulator respectively. If d is a square free integer, h_d is the number of ideal classes with respect to multiplication by elements with positive norms. Therefore, this h_d is slightly different from h_k of a number field of discriminant d even though they are closely related.

The problem of counting h_kR_k of quadratic fields k was first settled by Goldfeld–Hoffstein [17] and was slightly generalized by Datskovsky [9] from our viewpoint recently. Here, we quote Datskovsky's statement for the simplest case.

(0.1.2)
$$\sum_{\substack{0<-\Delta_k<x \\ [k:\mathbb{Q}]=2}} h_k \sim \frac{\zeta(2)}{3\pi} \prod_p (1-p^{-2}-p^{-3}+p^{-4}) x^{\frac{3}{2}},$$
$$\sum_{\substack{0<\Delta_k<x \\ [k:\mathbb{Q}]=2}} h_k R_k \sim \frac{\zeta(2)}{3} \prod_p (1-p^{-2}-p^{-3}+p^{-4}) x^{\frac{3}{2}},$$

where k runs through quadratic fields and Δ_k is the discriminant. Note that $\frac{\zeta(2)}{3} = \frac{\pi^2}{18}$. Therefore, (0.1.1) and (0.1.2) are very similar except for the difference between $\frac{1}{\zeta(3)}$ and $\prod_p(1-p^{-2}-p^{-3}+p^{-4})$.

Statements like (0.1.2) are the kind of density theorems we are looking for. We discuss the difference between (0.1.1) and (0.1.2) later in the introduction, and we go back to the space of binary quadratic forms again. The main ingredients of the above correspondence were the group $G = \mathrm{GL}(2)$ acting on the vector space V of binary quadratic forms, and the polynomial $\Delta(x) = x_1^2 - 4x_0x_2$ ($x = (x_0, x_1, x_2)$) which satisfies the property $\Delta(gx) = \det g\Delta(x)$ for $g \in \mathrm{GL}(2), x \in V$. Moreover, if we consider this vector space over an algebraically closed field, the generic point is a single G-orbit. The fundamental reason why one can prove results like (0.1.1), (0.1.2) is that we can use the Fourier analysis on the vector space V. Also when we consider the averages as (0.1.1) or (0.1.2), we can use the value of $\Delta(x)$ to average over. The fact that the generic point is a single orbit assures us that there is essentially one choice of such a polynomial.

Sato and Shintani introduced the notion of prehomogeneous vector spaces in [60] and generalized the situation as the above example. We now state the definition of prehomogeneous vector spaces from our viewpoint.

Let k be an arbitrary field. Let G be a connected reductive group, V a representation of G, and χ_V an indivisible non-trivial rational character of G, all defined over k.

Definition (0.1.3) *The triple (G, V, χ_V) is called a prehomogeneous vector space if the following two conditions are satisfied.*
(1) There exists a Zariski open orbit.
(2) There exists a polynomial $\Delta \in k[V]$ such that $\Delta(gx) = \chi'(g)\Delta(x)$ where χ' is a rational character and $\chi' = \chi_V^a$ for some positive integer a.

Note that if Δ_1, Δ_2 are two polynomials as in the above definition, there exist positive integers a, b such that $\frac{\Delta_1^a}{\Delta_2^b}$ is a G-invariant rational function. Since there exists an open orbit, this implies that $\frac{\Delta_1^a}{\Delta_2^b}$ is a constant function. Therefore, for any

k-algebra R, the set $V_R^{ss} = \{x \in V_R \mid \Delta(x) \neq 0\}$ does not depend on the choice of Δ, and we call it the set of semi-stable points. A polynomial Δ which satisfies the property (2) is called a relative invariant polynomial. If V is an irreducible representation, the center of the image $G \to \mathrm{GL}(V)$ has split rank one by Schur's lemma. Therefore the choice of χ_V is unique. So we call (G, V) a prehomogeneous vector space also.

For the space of binary quadratic forms, let $G = \mathrm{GL}(1) \times \mathrm{GL}(2)$, and $\chi_V(t, g) = t \det g$ for $g = (t, g)$. Then (G, V, χ_V) is a prehomogeneous vector space in the above sense.

Before we start the discussion on prehomogeneous vector spaces, let us make one more historical remark.

There is no doubt that Gauss was the first mathematician who recognized the group theoretic approach to number theory. But one particular prehomogeneous vector space had appeared already in the eleventh century.

The solution of cubic and quartic equations by radicals has been known for a long time. But before the solution was found, there was a poet-mathematician Omar Khayyam in medieval Persia who worked on this problem. He did not think it was possible to solve cubic equations by radicals, and instead he tried to express the solutions to cubic equations geometrically. For example, the solution of the equation $x^3 = N$ can be realized as the intersection of two parabolas $y = x^2, y^2 = Nx$. After the solution by radicals was found, his work has long been forgotten. However, it is surprisingly related to the theory of prehomogeneous vector spaces.

What makes the space of binary quadratic forms so interesting is that we can associate a quadratic field to a generic point of the vector space. More precisely, if G is $\mathrm{GL}(1) \times \mathrm{GL}(2)$, V is the space of binary quadratic forms and $\chi_V(t, g) = t \det g$, there is a map from $G_{\mathbb{Q}} \setminus V_{\mathbb{Q}}^{ss}$ to the set of isomorphism classes of fields of degree less than or equal to 2 over $\mathbb{Q}$. This map is clearly surjective, and this surjectivity is the reason why we count the class number of all the quadratic fields in (0.1.2).

Now, let us consider the group $G = \mathrm{GL}(3) \times \mathrm{GL}(2)$, and the vector space V of pairs of ternary quadratic forms. If we define $\chi_V(g_1, g_2) = (\det g_1)^4 (\det g_2)^3$, the triple (G, V, χ_V) is a prehomogeneous vector space (see [59] or Chapter 8). If $x = (Q_1, Q_2) \in V_k$ and Q_1, Q_2 are ternary quadratic forms, we can consider the set $\mathrm{Zero}(x) = \{v \in \mathbb{P}^2 \mid Q_1(v) = Q_2(v) = 0\}$. We call $\mathrm{Zero}(x)$ the zero set of x. We will show in Chapter 8 that (G, V, χ_V) is a prehomogeneous vector space and V^{ss} consists of points whose zero sets are four distinct points in $\mathbb{P}^2$. It is easy to see that the field generated by the residue fields of points in $\mathrm{Zero}(x)$ is a splitting field of a quartic equation. Now the question is if we can get all such fields from pairs of ternary quadratic forms. This is easy because any quartic equation $x^4 + a_1 x^3 + a_2 x^2 + a_3 x + a_4 = 0$ can be written as an intersection of two conics as follows.

$$y = x^2, y^2 + a_1 xy + a_2 x^2 + a_3 x + a_4 = 0.$$

But this is what Omar Khayyam did about 900 years ago, and he essentially proved the surjectivity of the map from $G_{\mathbb{Q}} \setminus V_{\mathbb{Q}}^{ss}$ to the isomorphism classes of splitting fields of quartic equations. For the works of Omar Khayyam, the reader should see [74]. The analytic theory of this prehomogeneous vector space is the main topic of this book, and we handle the global zeta function for this case in Part IV.

§0.2 The classification

In this section, we discuss the classification of irreducible prehomogeneous vector spaces over an algebraically closed field. Throughout this section, k is an algebraically closed field of characteristic zero.

First, we show that from a given prehomogeneous vector space, we can make infinitely many prehomogeneous vector spaces which are essentially the same as the original prehomogeneous vector space.

Let G be a reductive group, and V a representation of G. Suppose that the dimension of V is n. For an integer $0 < m < n$, we consider two representations $(G\times \mathrm{GL}(m), V\otimes k^m)$, $(G\times\mathrm{GL}(n-m), V\otimes k^{n-m})$. Then generic $\mathrm{GL}(m)_k$-orbits of $V_k\otimes k^m$ correspond bijectively with generic $\mathrm{GL}(n-m)_k$-orbits of $V_k \otimes k^{n-m}$ because the Grassmann variety of m planes in V can be identified with the Grassmann variety of $n-m$ planes in V. Therefore, $(G\times\mathrm{GL}(m), V\otimes k^m)$ is a prehomogeneous vector space if and only if $(G\times \mathrm{GL}(n-m), V\otimes k^{n-m})$ is a prehomogeneous vector space, and the sets of generic orbits coincide. If two prehomogeneous vector spaces are related in this way, we identify two such representations and consider the equivalence relation determined by this identification. If the dimension of (G, V) is the smallest among prehomogeneous vector spaces which are equivalent to (G, V), we say that (G, V) is reduced. Also we identify two prehomogeneous vector spaces $(G, V), (G', V)$ if the images of G, G' in $\mathrm{GL}(V)$ are the same.

Sato and Kimura proved in [59] that the following is the list of all the irreducible reduced prehomogeneous vector spaces.

(1) $G = \mathrm{GL}(n) \times H$, $V = M(n,n)_k$ where $H \subset \mathrm{GL}(n)$ is any reductive subgroup such that the k^n is an irreducible representation of H.

(2) $G = \mathrm{GL}(1) \times \mathrm{GL}(n)$, $V = \mathrm{Sym}^2 k^n$.

(3) $G = \mathrm{GL}(1) \times \mathrm{GL}(2n)$, $V = \wedge^2 k^{2n}$.

(4) $G = \mathrm{GL}(1) \times \mathrm{GL}(2)$, $V = \mathrm{Sym}^3 k^2$.

(5), (6), (7) $G = \mathrm{GL}(1) \times \mathrm{GL}(n)$, $V = \wedge^3 k^n$ where $n = 6, 7, 8$.

(8) $G = \mathrm{GL}(3) \times \mathrm{GL}(2)$, $V = \mathrm{Sym}^2 k^3 \otimes k^2$.

(9) $G = \mathrm{GL}(6) \times \mathrm{GL}(2)$, $V = \wedge^2 k^6 \otimes k^2$.

(10), (11) $G = \mathrm{GL}(n) \times \mathrm{GL}(5)$, $V = k^n \otimes \wedge^2 k^5$ where $n = 3, 4$.

(12) $G = \mathrm{GL}(3) \times \mathrm{GL}(3) \times \mathrm{GL}(2)$, $V = k^3 \otimes k^3 \otimes k^2$.

(13) $G = \mathrm{GSp}(2n) \times \mathrm{GL}(2m)$, $V = k^{2n} \otimes k^{2m}$ where $n \geq 2m \geq 2$.

(14) $G = \mathrm{GL}(1) \times \mathrm{GSp}(6)$, V is a 14 dimensional representation of G.

(15) $G = \mathrm{GO}(n) \times \mathrm{GL}(m)$, $V = k^n \otimes k^m$ where $n \geq 3$, $\frac{n}{2} \geq m \geq 1$.

(16), (17), (18) $G = \mathrm{GSpin}(7) \times \mathrm{GL}(n)$, $V = \mathrm{spin}_7 \otimes k^n$ where $n = 1, 2, 3$ and spin_7 is the spin representation.

(19), (22) $G = \mathrm{GSpin}(n)$, $V = \mathrm{spin}_n$ where $n = 9, 11$.

(20), (21) $G = \mathrm{GSpin}(10) \times \mathrm{GL}(n)$, $V = \mathrm{halfspin}_{10} \otimes k^n$ where $\mathrm{halfspin}_{10}$ is the halfspin representation and $n = 2, 3$.

(23), (24) $G = \mathrm{GL}(1) \times \mathrm{GSpin}(n)$, $V = \mathrm{halfspin}_n$ where $n = 12, 14$.

(25), (26) $G = G_2 \times \mathrm{GL}(n)$, $V = k^7 \otimes k^n$ where k^7 is a representation of G_2 and $n = 1, 2$.

(27), (28) $G = E_6 \times \mathrm{GL}(n)$, $V = k^{27} \otimes k^n$ where k^{27} is a representation of E_6 and $n = 1, 2$.

(29) $G = \mathrm{GL}(1) \times E_7$, V is a 56 dimensional representation of E_7

(30) $G = \mathrm{GSp}(2n) \times \mathrm{GO}(3)$, $V = k^{2n} \otimes k^3$.

The cases (1)–(29) are what we call regular prehomogeneous vector spaces. For the definition of the regularity, the reader should see [59]. Even though it does not make any difference over an algebraically closed field, we have included the GL(1) factor in (2)–(5) etc. and used groups like $\mathrm{GSp}(2n), \mathrm{GO}(n)$ instead of $\mathrm{Sp}(2n), \mathrm{O}(n)$ etc., because it is more natural number theoretically. Most of these representations are what we call prehomogeneous vector spaces of parabolic type classified by Rubenthaler in his thesis [52]. This is the kind of prehomogeneous vector spaces which one can construct from parabolic subgroups of reductive groups as follows.

Let G be a reductive group, and $P = MU$ a standard parabolic subgroup where M is the Levi component and U is the unipotent radical. The reductive part M acts on U by conjugation, and therefore on $V = U/[U,U]$ also. Since V can be considered as a vector space, we have a representation of a reductive group M. Vinberg [75] proved that there is a Zariski open orbit. Therefore, if there exists a relatively invariant polynomial, (M, V) is a prehomogeneous vector space by choosing a relative invariant polynomial and is called a prehomogeneous vector space of parabolic type.

For example, if we consider the Siegel parabolic subgroup P of $\mathrm{GSp}(2n)$, $M = \mathrm{GL}(1) \times \mathrm{GL}(n)$ and V is the space of quadratic forms in n variables. If G is a type C_n group etc., we say that (M, V) is of type C_n etc. Then (2) is C_n type, (3) is D_{2n} type, (4) is G_2 type, (5), (6), (7) are of E_6, E_7, E_8 types, (8) is F_4 type, (9) is E_7 type, (10), (11) are E_7, E_8 types, (12) is E_6 type, (13) is C_{n+m} type, (14) is F_4 type, (15) is B, D type, (16) is F_4 type, (20), (23), (25), (27) are E_7 type, (21), (24), (26), (28) are E_8 type (29) is E_8 type. (1) is not always of parabolic type. (17), (18) (19), (22), (25), (26) are not in Table 1 [52, pp. 35–38].

For the details on prehomogeneous vector spaces of parabolic type, the reader should see [52].

§0.3 The global zeta function

In this section, we discuss the meromorphic continuation and the functional equation of the zeta function, restricting ourselves to irreducible prehomogeneous vector spaces (G, V, χ_V) for simplicity. The reader should see §3.1 for the general definition of the zeta function. For the rest of this section, k is a number field.

For simplicity, we assume that there exists a one dimensional split torus $T_0 \cong \mathrm{GL}(1)$ in the center of G acting on V by the ordinal multiplication by $t^{e_0} \in \mathrm{GL}(1)$ and $\chi_V(t) = t^e$ for $t \in T_0$ where $e_0, e > 0$ are positive integers. Let Δ be a relative invariant polynomial, and d the degree of Δ. Then $|\Delta(gx)| = |\chi_V(g)|^{\frac{e_0 d}{e}} |\Delta(x)|$. Let N be the dimension of V.

We assume that the representation $G \to \mathrm{GL}(V)$ is faithful. Therefore, in terms of the list in §0.2, we are considering $(G/\widetilde{T}, V)$ where $\widetilde{T}$ is the kernel of the homomorphism $G \to \mathrm{GL}(V)$. We fix a Haar measure dg on $G_{\mathbb{A}}$. Moreover, we assume that dg is of the form $dg = \prod_v dg_v$ where dg_v is a Haar measure on G_{k_v} for $v \in \mathfrak{M}$. Let $L \subset V_k^{\mathrm{ss}}$ be a G_k-invariant subset. For $\Phi \in \mathscr{S}(V_{\mathbb{A}})$ and a complex variable s, we define

$$Z_L(\Phi, s) = \int_{G_{\mathbb{A}}/G_k} |\chi_V(g)|^{\frac{e_0 s}{e}} \sum_{x \in L} \Phi(gx) dg, \tag{0.3.1}$$