NONLINEAR PHENOMENA AND COMPLEX SYSTEMS

DEAN J. DRIEBE

Fully Chaotic Maps and Broken Time Symmetry

Kluwer Academic Publishers

Fully Chaotic Maps and Broken Time Symmetry

by

Dean J. Driebe

The University of Texas at Austin,
Texas, U.S.A.
and
International Solvay Institutes for Physics and Chemistry,
Brussels, Belgium

KLUWER ACADEMIC PUBLISHERS

DORDRECHT / BOSTON / LONDON

A C.I.P. Catalogue record for this book is available from the Library of Congress.

ISBN 0-7923-5564-4

Published by Kluwer Academic Publishers,
P.O. Box 17, 3300 AA Dordrecht, The Netherlands

Sold and distributed in North, Central and South America
by Kluwer Academic Publishers,
101 Philip Drive, Norwell, MA 02061, U.S.A.

In all other countries, sold and distributed
by Kluwer Academic Publishers
P.O. Box 322, 3300 AH Dordrecht, The Netherlands

Printed on acid-free paper

Printed in Belgium

Fully Chaotic Maps and Broken Time Symmetry

Nonlinear Phenomena and Complex Systems

VOLUME 4

The Centre for Nonlinear Physics and Complex Systems (CFNL), Santiago, Chile, and Kluwer Academic Publishers have established this series devoted to nonlinear phenomena and complex systems, which is one of the most fascinating fields of science today, to publish books that cover the essential concepts in this area, as well as the latest developments. As the number of scientists involved in the subject increases continually, so does the number of new questions and results. Nonlinear effects are essential to understand the behaviour of nature, and the methods and ideas introduced to treat them are increasingly used in new applications to a variety of problems ranging from physics to human sciences. Most of the books in this series will be about physical and mathematical aspects of nonlinear science, since these fields report the greatest activity.

The titles published in this series are listed at the end of this volume.

Foreword

I am very pleased and privileged to write a short foreword for the monograph of Dean Driebe: *Fully Chaotic Maps and Broken Time Symmetry.* Despite the technical title this book deals with a problem of fundamental importance. To appreciate its meaning we have to go back to the tragic struggle that was initiated by the work of the great theoretical physicist Ludwig Boltzmann in the second half of the 19th century.

Ludwig Boltzmann tried to emulate in physics what Charles Darwin had done in biology and to formulate an evolutionary approach in which past and future would play different roles. Boltzmann's work has lead to innumerable controversies as the laws of classical mechanics (as well as the laws of quantum mechanics) as traditionally formulated imply symmetry between past and future. As is well known, Albert Einstein often stated that "Time is an illusion". Indeed, as long as dynamics is associated with trajectories satisfying the equations of classical mechanics, explaining irreversibility in terms of trajectories appears, as Henri Poincaré concluded, as a logical error. After a long struggle, Boltzmann acknowledged his defeat and introduced a probability description in which all microscopic states are supposed to have the same a priori probability. Irreversibility would then be due to the imperfection of our observations associated only with the "macroscopic" state described by temperature, pressure and other similar parameters. Irreversibility then appears devoid of any fundamental significance.

However today this position has become untenable. Nonequilibrium physics has shown that the flow of time plays an essential constructive role as it leads to nonequilibrium structures; often called "dissipative structures" to distinguish them from equilibrium structures such as crystals. It becomes absurd to imagine that we, through our approximations, are at the origin of the arrow of time found at all levels of observation. A time-reversible world would also be a world we could not learn how to describe as every experiment implies a difference between past and future.

What then is the way out? Gradually I was driven to the conclusion that the traditional formulations of classical and quantum mechanics have to be extended to include time symmetry breaking.[1] This, however, requires the construction of a new mathematical formulation which reduces to the usual formulation of classical or quantum mechanics in simple situations.

---

[1] I. Prigogine, *From Being to Becoming* (W.H. Freeman and Company, New York, 1980); I. Prigogine and I. Stengers, *Order Out of Chaos* (Bantam, New York, 1983); I. Prigogine, *The End of Certainty* (The Free Press, New York, 1997).

While the physical ideas have been clear for a considerable length of time[2] a precise mathematical formulation has emerged only during the past few years. The simplest example where we can at present extend the laws of classical dynamics is precisely deterministic chaos, the subject of this book.

The essential point is that dynamics can be formulated either on the level of individual trajectories or in terms of ensembles as introduced by Gibbs and Einstein in their fundamental work on thermodynamics. Traditionally ensembles were associated with ignorance. It was always assumed that from the dynamical point of view the two descriptions are equivalent. However – and that is the most interesting conclusion described in this book – this is not so for chaotic systems. The probabilistic description in terms of ensembles is properly formulated with operator theory extended to generalized functional spaces. Then the ensemble description leads to new solutions irreducible to the usual description in terms of trajectories. That is what we mean by the "extension" of classical dynamics. Thanks to this recent development, we may consider that Boltzmann's time paradox has found its natural resolution. It is no more necessary to make any reference to extra-dynamical features such as coarse graining or environmental noise as has been done in the past.

Chaotic maps are only one example where the individual description in terms of trajectories and the ensemble description lead to different formulations. Thermodynamic systems form another class. This class has been studied elsewhere.[3] Dean Driebe's presentation is limited to chaotic maps but even so it makes fascinating reading as it shows how to solve a long-standing paradox, which has been hotly debated for over a hundred years.

Dean Driebe is especially well-prepared to write this monograph as he has made a number of original contributions to the subject. I am sure that his book will be of great interest to all scientists, philosophers and engineers who are interested in the perennial questions of the limits of determinism and the meaning of time.

Ilya Prigogine

---

[2] I. Prigogine, *Nonequilibrium Statistical Mechanics* (Wiley-Interscience, NewYork, 1962).

[3] T. Petrosky and I. Prigogine, "Poincaré resonances and the extension of classical dynamics," *Chaos, Solitons & Fractals* **7**, 441 (1996); "The Liouville space extension of quantum mechanics," in *Advances in Chemical Physics*, **99**, (John Wiley, New York, 1997).

Preface

This book originated from notes for a series of lectures given by the author at the University of Chile in December 1994. The purpose of the lectures was to present some of the recent work, mainly of the groups directed by I. Prigogine in Austin and Brussels, on the time evolution of densities in chaotic systems and its relevance to the problem of irreversibility. The emphasis was on the construction of new spectral decompositions of time evolution operators in generalized functional spaces. These decompositions allow for a detailed study of nonequilibrium processes and an understanding of time-symmetry breaking. The approach realizes part of the goal of the Prigogine group to understand irreversibility as an intrinsic property of unstable dynamical systems.

The book deals only with fully chaotic maps, where complete, exact spectral decompositions have been obtained. Besides the intrinsic interest of these systems – even if they don't display generic behavior from a physical point of view – they elucidate the main assertion that dynamical instability is the root of irreversibility. Several advances have been made in the last couple of years, mainly in explicit results obtained for a variety of one-dimensional maps and the discovery of an unexpectedly rich variety of spectra found in a class of simple one-dimensional piecewise-linear maps. These new results, some of which have not yet been published, have been included so that the book gives an up-to-date presentation. The purpose though has not been to give an exhaustive review of the subject but to write an introduction for students and research workers who want to know how the generalized spectral decompositions are obtained and the range of systems that have been considered. The presentation does not strive for mathematical rigor, rather it emphasizes ideas, results and calculational tools. The reader interested in mathematical details may refer to the literature cited as well as the forthcoming monograph by I. Antoniou.

The first two chapters essentially cover background material. In Chapter 1 the motivation for the approach used is discussed as pertinent to the dynamical understanding of irreversibility. An introduction to the concept of probability densities in the phase space of a chaotic system is given and the hierarchy of types of behavior in phase space is presented. In Chapter 2 a more detailed discussion of nonequilibrium statistical mechanics of chaotic maps is given and time correlation functions of observables is discussed in the context of how their behavior reflects on the spectral properties of the time evolution operator of the system.

The heart of the book begins in Chapter 3 where the simple one-dimensional

Bernoulli map is studied and the construction of the spectral decomposition of its Frobenius–Perron operator is given in detail. In this and the following chapter one-dimensional systems with non-invertible trajectory dynamics are considered. These systems are irreversible from the beginning but they share many key features with the invertible systems considered later in the book. In Chapter 4 a variety of maps of the unit interval are discussed. An algebraic technique utilizing symmetry is used to obtain the explicit spectral decomposition of some maps, including the well-known tent map. Also, a map with a fractal repellor is considered and the decomposition of topologically conjugate maps is discussed and applied to obtain the decomposition of the logistic map with unit height.

Chapter 5 is devoted to the baker transformation as the paradigm system to elucidate the time-symmetry breaking of the generalized spectral decomposition. The trajectory dynamics of the baker map is invertible so the associated time evolution operators for densities or observables are unitary in a Hilbert space setting. The group evolution in Hilbert space splits into two distinct semigroups in the generalized representation. In Chapter 6 a model system of deterministic diffusion is considered. Transport properties are identified in the exact spectrum of the full time evolution operator of the system and quite interesting generalized eigenstates with fractal properties appear.

There are several appendices collected at the end of the book. This material expands on some topics discussed in the main text and fills in details of some of the calculations. At the end of each chapter appear bibliographical notes with comments on what can be found in the relevant books and papers. No attempt has been made to be exhaustive in the referencing. Also, in the text itself there do not generally appear specific reference citations. I feel that this is appropriate for a book presentation.

I am grateful to Enrique Tirapegui for iniviting me to Santiago to give the lectures and write this book. My utmost gratitude goes to I. Prigogine for his support and encouragement and for his kindness in writing the foreword. I thank Hiroshi H. Hasegawa from whom I learnt many of the methods presented in this book. Many of the recent results presented in Chapter 4 have been obtained in collaboration with Gonzalo Ordóñez. My progress in the subject benefitted over the years from discussions with Ioannis Antoniou, Oscar Bandtlow, Francisco Bosco, Pierre Gaspard, Donal MacKernan, Bill Saphir and Shuichi Tasaki. I also acknowledge Irene Burghardt, David Daems, Brian LaCour and Suresh Subbiah for their comments on draft versions of the manuscript. Thanks is due Annie Harding for typing the original manuscript and David Leonard for assistance in preparing the figures.

Nonlinear Phenomena and Complex Systems

1. E. Tirapegui and W. Zeller (eds.): *Instabilities and Nonequilibrium Structures V*. 1996 ISBN 0-7923-3992-4
2. E. Goles and S. Martínez (eds.): *Dynamics of Complex Interacting Systems*. 1996 ISBN 0-7923-4173-2
3. E. Goles and S. Martínez (eds.): *Cellular Automata and Complex Systems*. 1999 ISBN 0-7923-5512-1
4. D. J. Driebe: *Fully Chaotic Maps and Broken Time Symmetry*. 1999 ISBN 0-7923-5564-4

KLUWER ACADEMIC PUBLISHERS – DORDRECHT / BOSTON / LONDON

Contents

Chapter 1

Chaos and Irreversibility

In contrast to the manifest irreversibility of nearly all systems we observe in nature, the basic dynamical laws of physics, be they classical or quantum, are time reversible. It has been a long-standing problem in physics to reconcile these two facts. The recent realization that unstable or chaotic dynamical systems are the most typical in nature has given fresh insight into this problem. We begin in this chapter with an informal discussion of the relation of unstable dynamics to the problem of irreversibility. We then discuss some general aspects of the evolution of probability densities in chaotic systems.

1.1 Irreversibility and Complexity

Traditional approaches to the explanation of irreversibility have always included extra-mechanical elements, such as coarse graining, which are difficult to justify and introduce subjectivity into the description of the time evolution of systems. These extra-mechanical elements have been considered necessary because the evolution laws for trajectories (or wavefunctions) are time reversible and the operators describing the evolution of ensembles are unitary so that time-oriented eigenmodes cannot be obtained in regular functional spaces.

The main idea of the Prigogine school of nonequilibrium statistical mechanics is to extend the formulation of the laws of dynamics to include irreversiblity on the fundamental level. This goal was inspired by the realization that irreversible processes are ubiquitous in nature and play a constructive role on many levels. Quoting from Prigogine's book *From Being to Becoming*: "Irreversibility corresponds not to some supplementary approximation introduced into the laws of dynamics but to an embedding of dynamics within a vaster formalism." Of course, this new formalism should reduce to the well-known reversible laws of dynamics for simple systems displaying, for example, periodic motion, such as harmonic oscillation or two-body attractive central force motion. Irreversibility should occur only if the system is sufficiently complex.

Complex behavior has traditionally been associated with systems of many degrees of freedom. As all statistical mechanics textbooks state: to solve Hamilton's equations for a system of N interacting particles, where N corresponds to a macroscopic sample so that $N \gtrsim 10^{23}$, is a hopelessly complicated task. In practice one then considers not the set of trajectories of the particles but a statistical description involving an ensemble distribution. From the point of view of describing the macroscopic behavior of a system, especially in equilibrium situations, this approach is generally valid since such behavior usually doesn't depend on details of the microscopic motions. If though one would like to understand the emergence of irreversible behavior from reversible microscopic dynamics, the bridge between these levels requires deeper considerations.

As is now well known, it is by no means necessary for a system to have many degrees of freedom to exhibit "complex" behavior. For a discrete time system, only one degree of freedom is necessary for chaotic behavior. Chaos means that the system displays "sensitivity to initial conditions." This means that two closely-spaced initial trajectories will spread exponentially in time under the dynamics.

Why should chaos be related to irreversibility? Let us quote David Ruelle from his book *Chaotic Evolution and Strange Attractors*. Speaking of chaos he states that "... the exponential growth of errors makes the time evolution self-independent from its past history and then nondeterministic in any practical sense." This statement certainly suggests that the applicability of time-reversiblity for a system whose evolution makes it self-independent from its past is in doubt. But Ruelle implies that this problem is only of a practical nature. For us, chaos leads to a formulation of dynamics transcending traditional theoretical approaches and allows for an understanding of irreversibility as an intrinsic property of chaotic systems.

A simple picture of chaotic evolution associated with diverging trajectories doesn't tell the whole story. In fact, the phase space of strongly chaotic systems in general has points (i.e., initial conditions) leading to regular (periodic) motion densely distributed among points leading to chaotic motion. (See Appendix A.1 for more on this.) This is in contrast to stable systems where different types of motion occur in finite regions of phase space with distinct boundaries between the regions.

The complex motion in chaotic systems naturally generates densities on the phase space. (If we follow the motion of a typical trajectory for a long time it will wander all over the chaotic region with the relative time spent in various parts of the region yielding a density, i.e., we consider ergodic systems – more about this later.) An initial nonequilibrium density may correspond to some uncertainty in the specification of the initial condition or may be thought of as representing an ensemble of systems with different initial conditions. A smooth density is supported on a finite region of phase space and so its evolution contains some non-local information that is missing in a point dynamical description.

For stable systems exhibiting regular motion the description of the dynamics in terms of the evolution of a density is generally reducible to a point density corresponding to an evolving trajectory. This is because in such a system the density may sample a region containing only one typical kind of trajectory behavior. For example, if for a pendulum we consider a smooth density supported on a region away from the separatrix we may reduce the density successively to a typical point in its support without a qualitative change in its behavior. But for chaotic systems the reduction from a smooth density corresponding to a statistical ensemble to a point density is wrought with inherent difficulties because the complex microstructure in phase space doesn't allow for an unambiguous limit. Quoting from Prigogine again: "Is this difficulty practical or theoretical in nature? I would support the view that this result has important theoretical and conceptual significance because it forces us to transgress the limits of a purely dynamical description." As we will see, the natural description for the time evolution in chaotic systems will be in terms of densities that are *irreducible* to trajectories. This representation of the time evolution operator involves generalized functional spaces. This will yield an intrinsically irreversible description for systems that nevertheless have time-reversible trajectory dynamics.

1.2 Densities in Chaotic Maps

In this book we are going to concentrate on the analysis of chaotic maps. Maps are discrete-time dynamical processes that may arise in several different contexts. A map may arise directly from the statement of a physical problem; for instance, a system that is kicked periodically and follows free motion between kicks. A map may also be constructed by taking a slice in phase space of a continuous-time system. Our interest in chaotic maps is that they are the simplest systems that have relevant features of chaotic

Hamiltonian systems. We use them as models to explore the dynamics and statistical mechanics of chaotic systems.

A map is specified by a dynamical law that determines how an initial point, x_0, evolves. (The dimension of the space of points to which x belongs may be greater than 1.) The map tells how to evolve one time step and to get t steps we apply an iterative procedure; thus, $x_t = S(x_{t-1}) = S(S(x_{t-2})) = \cdots = S_t(x_0)$, where S is the rule for the map. This procedure yields a trajectory for the system. There now exists a huge literature on trajectory dynamics in chaotic systems. The principal characterization of chaotic trajectory dynamics is given by the values of the positive Lyapunov exponents, which determine the rate of exponential spreading of nearby trajectories.

An alternative picture of the dynamics may be obtained from a statistical mechanics approach by considering an ensemble description. If we pick N initial points: $x_0^1, x_0^2, \ldots, x_0^N$ and apply the map to each point we obtain N new points: $x_1^1, x_1^2, \ldots, x_1^N$. A density, $\rho(x,t)$, at time step t, will describe this ensemble of N points if

$$\int_\Delta dx\, \rho(x,t) \simeq \frac{1}{N}\sum_{j=1}^{N} \chi_\Delta(x_t^j), \tag{1.1}$$

where

$$\chi_\Delta(y) = \begin{cases} 1, & y \in \Delta \\ 0, & y \notin \Delta \end{cases}$$

is the characteristic function of a small set Δ.

To determine the rule for the evolution of densities given the rule for points (i.e., trajectories) consider the evolution of a density corresponding to a trajectory. The point x_0 evolving to $S(x_0)$ after one iteration is equivalent to the singular density described by a Dirac delta function, $\delta(x-x_0)$, evolving to $\delta(x - S(x_0))$. The new density may be written in terms of the original one as

$$\delta(x - S(x_0)) = \int dx' \delta(x - S(x'))\delta(x' - x_0). \tag{1.2}$$

The evolution of a smooth density may be obtained then just by superposition. Since $\rho(x,t) = \int dx_0\, \delta(x - x_0)\rho(x_0,t)$ we obtain

$$\rho(x,t+1) = \int dx' \delta(x - S(x'))\rho(x',t) \equiv U\rho(x,t), \tag{1.3}$$

where we have defined the operator U, called the Frobenius–Perron operator, which evolves densities. Thus, in order to obtain the density at time t from an initial density at time $t = 0$ the Frobenius–Perron operator is applied t times as $\rho(x,t) = U^t\rho(x,0)$.

A very important fact we may notice immediately is that U is a linear operator. This is in contrast to the trajectory evolution which proceeds by an iterative process that is highly nonlinear. We may thus employ linear operator theory to study chaotic dynamics from the point of view of evolving densities. Even though the evolution of a density is determined by superposing trajectories, we will see later that we may nevertheless construct spectral decompositions of U that are irreducible to trajectories.

The behavior of trajectories and densities in a given system may be strikingly different. For a uniformly chaotic system a typical trajectory will wander around in the phase space in an apparently random fashion. Even in systems with non-invertible trajectory dynamics, a time series of the trajectory may look qualitatively similar forward or backward in time. In contrast, the evolution of a density will usually be obviously time-oriented and will often approach an attracting equilibrium state. These two types of behavior are illustrated in Figures 1.1 and 1.2 for the simple example of the one-dimensional dyadic Bernoulli map on the interval $[0,1)$. This system will be studied in detail in Chapter 3.

We note then a somewhat complementary character of trajectories versus densities. For a system with very chaotic or irregular trajectory behavior the density will normally show a quite regular behavior. On the other hand, for regular motion, such as periodic trajectories, the density behavior will generally mirror the trajectory behavior.

Discussions of the approach to equilibrium in simple chaotic systems are not very prevalent in the literature. In many places where the Frobenius–Perron operator is discussed it is used just to obtain the invariant density of the system. In fact, many authors discuss invariant densities and even show pictures of them generated numerically and comment how they were obtained "after neglecting transients." For us the so-called transients are our main interest.

1.3 Types of Behavior in Phase Space

The dynamics of a system in phase space may be classified according to its behavior over long times. A hierarchy of behavior may be identified and the Frobenius-Perron operator is useful for the classification. In the next chapter we will discuss in more detail how to do statistical mechanics of chaotic maps. For the present discussion it is only necessary to know that