Springer INdAM Series 4

Franco Brezzi
Piero Colli Franzone
Ugo Gianazza
Gianni Gilardi *Editors*

Analysis and Numerics of Partial Differential Equations

Springer

Franco Brezzi • Piero Colli Franzone •
Ugo Gianazza • Gianni Gilardi
Editors

Analysis and Numerics of Partial Differential Equations

Springer

Editors
Franco Brezzi
Istituto di Matematica Applicata e
Tecnologie Informatiche (IMATI)
CNR Pavia
Pavia, Italy

Ugo Gianazza
Dipartimento di Matematica "F. Casorati"
Università degli Studi di Pavia
Pavia, Italy

Piero Colli Franzone
Dipartimento di Matematica "F. Casorati"
Università degli Studi di Pavia
Pavia, Italy

Gianni Gilardi
Dipartimento di Matematica "F. Casorati"
Università degli Studi di Pavia
Pavia, Italy

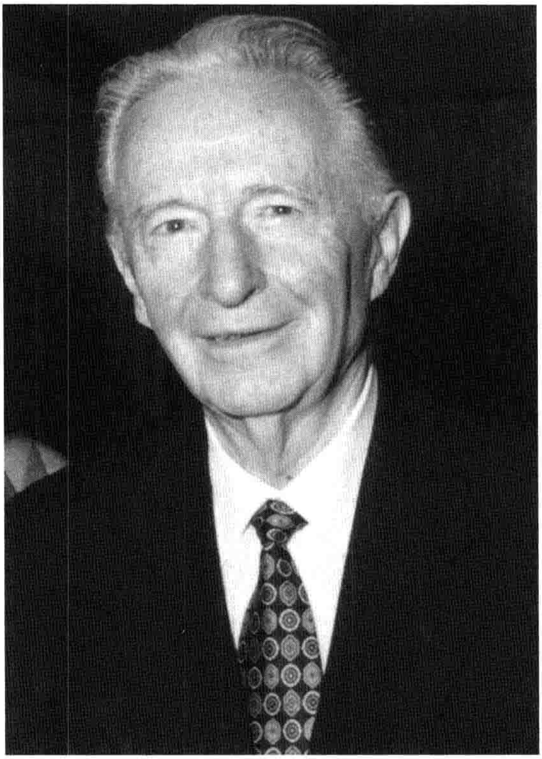
ISSN 2281-518X ISSN 2281-5198 (electronic)
Springer INdAM Series
ISBN 978-88-470-2591-2 ISBN 978-88-470-2592-9 (eBook)
DOI 10.1007/978-88-470-2592-9
Springer Milan Heidelberg New York Dordrecht London

Library of Congress Control Number: 2012951305

Printed on acid-free paper

Springer is part of Springer Science+Business Media (www.springer.com)

In memory of Enrico Magenes



Enrico Magenes (courtesy of the family)

Preface

On November 2nd 2010 Enrico Magenes passed away.

One year later, some of his friends, collaborators and former students organized a three-day conference, in order to celebrate his memory and give a first assessment of his deep influence on contemporary Mathematics. All the speakers were experts in the analysis and numerics of partial differential equations, who had directly interacted with Magenes, during his long career.

The present volume is a direct offshoot of that meeting, and it collects the main contributions offered in that occasion, properly revised and expanded. It consists of two parts: the first one gives a wide historical perspective of Magenes' work; the second one contains original research or survey papers, and shows how ideas, methods, and techniques introduced by Magenes and his collaborators still have an impact on the current research in Mathematics. As agreed between Springer and UMI (Unione Matematica Italiana), some of the papers appearing in the second part will be fully published on Bollettino UMI as well.

Although it is still too early to fully appreciate Magenes' legacy, nonetheless the volume is a first attempt to present a comprehensive survey of his activity in Mathematics. At the same time, from Magenes' peculiar point of view, it is a broad perspective of the research in partial differential equations and their applications developed in Italy in the period 1950–2000.

The editors are grateful to Francesca Bonadei of Springer Italy for the unique opportunity offered with the publication of this volume, and for her constant support during its preparation.

Pavia, Italy

Franco Brezzi
Piero Colli Franzone
Ugo Gianazza
Gianni Gilardi

List of Contributors

Luigi Ambrosio Scuola Normale Superiore, Pisa, Italy

Douglas N. Arnold School of Mathematics, University of Minnesota, Minneapolis, MN, USA

Claudio Baiocchi Dipartimento di Matematica "F. Casorati", Università di Pavia, Pavia, Italy; Dipartimento di Matematica "G. Castelnuovo", Università di Roma "La Sapienza", Rome, Italy

Katayun Barmak Department of Applied Physics and Applied Mathematics, Columbia University, New York, NY, USA

Andrea Bonito Department of Mathematics, Texas A&M University, College Station, TX, USA

Franco Brezzi Istituto di Matematica Applicata e Tecnologie Informatiche "E. Magenes", Consiglio Nazionale delle Ricerche, Pavia, Italy

Luis Caffarelli Department of Mathematics, University of Texas at Austin, Austin, TX, USA

Claudio Canuto Dipartimento di Scienze Matematiche, Politecnico di Torino, Turin, Italy

J. Manuel Cascón Departamento de Economía e Historia Económica, Universidad de Salamanca, Salamanca, Spain

Piero Colli Franzone Dipartimento di Matematica "F. Casorati", Università di Pavia, Pavia, Italy

Eva Eggeling Visual Computing, Fraunhofer Austria Research GmbH, Graz, Austria

Maria Emelianenko Department of Mathematical Sciences, George Mason University, Fairfax, VA, USA

Yekaterina Epshteyn Department of Mathematics, The University of Utah, Salt Lake City, UT, USA

Mario Ferrari Dipartimento di Matematica "F. Casorati", Università di Pavia, Pavia, Italy

Giuseppe Geymonat LMS, Laboratoire de Mécanique des Solides UMR 7649, École Polytechnique, Palaiseau Cedex, France

Ugo Gianazza Dipartimento di Matematica "F. Casorati", Università di Pavia, Pavia, Italy

Nicola Gigli Université de Nice, Mathématiques, Nice, France

David Kinderlehrer Department of Mathematical Sciences, Carnegie Mellon University, Pittsburgh, PA, USA

Toni Lassila MATHICSE-CMCS, Modelling and Scientific Computing, École Polytechnique Fédérale de Lausanne, Lausanne, Switzerland

Yvon Maday Laboratoire Jacques-Louis Lions, UMR 7598, UPMC Univ Paris 06, Paris, France; Division of Applied Mathematics, Institut Universitaire de France and Brown University, Providence, RI, USA

Andrea Manzoni SISSA Mathlab, SISSA—International School for Advanced Studies, Trieste, Italy

Pedro Morin Instituto de Matemática Aplicada del Litoral (IMAL), Güemes 3450 and Departamento de Matemática, Facultad de Ingeniería Química, Universidad Nacional del Litoral, Santa Fe, Argentina

Umberto Mosco Department of Mathematical Sciences, Worcester Polytechnic Institute, Worcester, MA, USA

Olga Mula Laboratoire Jacques-Louis Lions, UMR 7598, UPMC Univ Paris 06, Paris, France; CEA Saclay—DEN/DANS/DM2S/SERMA/LLPR, Gif-Sur-Yvette Cedex, France

Ricardo H. Nochetto Department of Mathematics and Institute for Physical Science and Technology, University of Maryland, College Park, MD, USA

Alfio Quarteroni MATHICSE-CMCS, Modelling and Scientific Computing, École Polytechnique Fédérale de Lausanne, Lausanne, Switzerland; MOX, Modeling and Scientific Computing, Dipartimento di Matematica, Politecnico di Milano, Milan, Italy

Gianluigi Rozza MATHICSE-CMCS, Modelling and Scientific Computing, École Polytechnique Fédérale de Lausanne, Lausanne, Switzerland; SISSA Mathlab, SISSA—International School for Advanced Studies, Trieste, Italy

Giuseppe Savaré Dipartimento di Matematica "F. Casorati", Università di Pavia, Pavia, Italy

Richard Sharp Microsoft Corporation, Redmond, WA, USA

Shlomo Ta'asan Department of Mathematical Sciences, Carnegie Mellon University, Pittsburgh, PA, USA

Enrico Valdinoci Dipartimento di Matematica "F. Enriques", Università di Milano, Milan, Italy; Istituto di Matematica Applicata e Tecnologie Informatiche "E. Magenes", Consiglio Nazionale delle Ricerche, Pavia, Italy

Marco Verani MOX-Dipartimento di Matematica, Politecnico di Milano, Milan, Italy

Claudio Verdi Dipartimento di Matematica "F. Enriques", Università di Milano, Milan, Italy

Augusto Visintin Dipartimento di Matematica, Università di Trento, Trento, Italy

Contents

Part I
A Historical Perspective

Personal Memories

Franco Brezzi

Abstract Few personal memories about Enrico Magenes as scientific mentor, chairman of the Institute of Numerical Analysis, President of the Italian Mathematical Union.

Enrico Magenes taught me A LOT, both as a mathematician and as a man.

This is not obviously the place to tell what I learned in Mathematics. It is a fact, however, that I tried to learn, at least in part, many of his human gifts: his commitment at work, his sincerity, his love for the talent, his warm-heartedness, a lot of things. Maybe the most important thing that I tried to learn from him, was "not to hate." As a matter of fact, I saw him a lot of times losing patience, but I never had the impression he hated anybody. There were things he disapproved of. There were persons he did not think much of. However, he never hated anybody, not even those, who had put him in a concentration camp.

For me, this was particularly important, and I did my best, in order to learn it.

Under a more general point of view, for us, his students, he was an example and a stimulus. In my opinion, his best scientific talent was one of those gifts, which are not frequently extolled. However, still in my opinion, it is perhaps the most important talent in a mentor: the ability to recognize the important problems. He could sense the scientific directions, that would generate a lot of important developments, and those that would extinguish after a couple of papers. It is in this way, with extreme far-sightedness, that he could open new ways, combining mathematical rigor with interest for applications, and starting a lot of fruitful collaborations with engineers, biologists and physicians. These are things, that at the time were almost revolutionary.

As a matter of fact, he never forced anybody to work on a specific topic. He just limited himself to suggesting the problems he considered important enough to be solved. All of us were obviously EXTREMELY careful in following his advices.

F. Brezzi (✉)
Istituto di Matematica Applicata e Tecnologie Informatiche "E. Magenes", Consiglio Nazionale delle Ricerche, via Ferrata 1, 27100, Pavia, Italy
e-mail: brezzi@imati.cnr.it

F. Brezzi et al. (eds.), *Analysis and Numerics of Partial Differential Equations*,
Springer INdAM Series 4, DOI 10.1007/978-88-470-2592-9_1,

I was his (unworthy) successor in two positions, which I consider important: the chairmanship of the Institute of Numerical Analysis of the Italian Research Council (CNR) (later on merged in the current Institute of Applied Mathematics and Information Technologies, still of the CNR) which I took in 1992, immediately after he finished his term, and the Presidency of the Italian Mathematical Union, which I took thirty years later (he finished his term in 1976, I started mine in 2006).

In the first instance, notwithstanding my commitment, the comparison was merciless. I try and console myself, telling me that nobody could possibly rival his enthusiasm, his rigor, his vision about Mathematics, and his humanity. Moreover, I tell myself that few would have been able, so to speak, to lose by a narrow margin. Anyway, the comparison was really hard, and I was lucky that he was there, always at disposal for suggestions, advices, warnings. As a matter of fact, also through the activities of the Institute (which originally was called Laboratory), Magenes could give birth to a large part of the Italian Applied Mathematics (in particular, Numerical Analysis), placing it among the highest ranking international positions: just preserving this ranking was very difficult.

The Italian Mathematical Union (UMI) owes him a lot, not only for the work done as a member of the Scientific Committee (from 1967 to 1979) and then as President in difficult and stormy times (from 1973 to 1976). For UMI Magenes was always a stimulus, a source of ideas and initiatives, and at the same time a balanced and wise presence, witness of a clear and solid vision, of what was important, and what not. Even when he did not hold any institutional office, at national level, all the same he was a reference point, somebody who could encourage or warn, a sort of Guardian Angel.

With him a large piece of history passes away, an important piece, not only for Mathematics. Good Bye Enrico!

Some Aspects of the Research of Enrico Magenes in Partial Differential Equations

Giuseppe Geymonat

Abstract The author traces the initial stage of Enrico Magenes's research, with a particular emphasis on his work in Partial Differential Equations. The very fruitful collaborations with G. Stampacchia and J.-L. Lions are clearly presented.

1 The Beginnings in Modena

The first researches of Enrico Magenes in Partial Differential Equations date to 1952, [14, 15] (and in the same year he became professor of Mathematical Analysis at the University of Modena). Their argument is the application to the heat equation of a method that in the Italian School is called "Picone's Method". The basic idea of the method is to transform the boundary value problem into a system of integral equations of Fischer-Riesz type. This idea was introduced by Picone around 1935 and then deeply applied by Amerio, Fichera, and many others to elliptic equations. For simplicity, we present the method in the simplest case of a non-homogeneous Dirichlet problem in a smooth, bounded domain $\Omega \subset \mathbb{R}^N$:

$$A(u) = f \quad \text{in } \Omega, \qquad \gamma_0 u = g \quad \text{on } \Gamma := \partial\Omega \tag{1}$$

where $A(u)$ is a second order linear elliptic operator with smooth coefficients and $\gamma_0 u$ denotes the trace of u on Γ; for simplicity one can also assume that uniqueness holds true for this problem. Let $A^\star$ be the formal adjoint of A and let $\frac{\partial}{\partial \nu}$ denote the so-called co-normal derivative (when $A = \Delta$, then $A^\star = \Delta$ and $\nu = \mathbf{n}$, the outgoing normal to Γ). The Green formula states

$$\int_\Omega A(u) w\,dx - \int_\Omega u A^\star(w)\,dx = \int_\Gamma \left(\frac{\partial u}{\partial \nu} w - u \frac{\partial w}{\partial \nu}\right) d\Gamma. \tag{2}$$

Hence, if one knows a sequence w_n of smooth enough functions, such that $A^\star(w_n)$ and $\gamma_0(w_n)$ both converge, then, thanks to (2), *the determination of the vector*

G. Geymonat (✉)
LMS, Laboratoire de Mécanique des Solides UMR 7649, École Polytechnique, Route de Saclay, 91128 Palaiseau Cedex, France
e-mail: giuseppe.geymonat@lms.polytechnique.fr

F. Brezzi et al. (eds.), *Analysis and Numerics of Partial Differential Equations*,
Springer INdAM Series 4, DOI 10.1007/978-88-470-2592-9_2,

$(u, \frac{\partial u}{\partial \nu})$ with u the solution of (1) *is reduced to the solution of a system of linear equations of Fischer-Riesz type.*

The difficulty was naturally to find such a sequence, to prove its completeness (in a suitable functional space) and hence the space where the corresponding system is solvable and so the problem (1).

Following Amerio [1], let the coefficients of A be smoothly extended to a domain $\widehat{\Omega} \supset \Omega$ and for every fixed $R \in \widehat{\Omega}$ let $F(P,R)$, as a function of P, be the fundamental solution of $A^\star(w) = 0$ (moreover, such a fundamental function can be chosen so that, as a function of R, it also satisfies $A(u) = 0$). Then from (1) and (2) it follows that for every $Q \in \widehat{\Omega} \setminus \Omega$

$$0 = \int_\Gamma \left(u(x) \frac{\partial F(x,Q)}{\partial \nu} - \frac{\partial u(x)}{\partial \nu} F(x,Q) \right) d\Gamma - \int_\Omega f(x) F(x,Q) dx. \qquad (3)$$

This equation gives a necessary compatibility condition between $\gamma_0 u$ and $\frac{\partial u(x)}{\partial \nu}$. Moreover, if φ_n is a sequence of "good" functions defined in $\overline{\widehat{\Omega} \setminus \Omega}$, then one can take $w_n(P) = \int_{\overline{\widehat{\Omega} \setminus \Omega}} \varphi_n(x) F(P,x) dx$. Two problems remain:

(i) the choice of the sequence φ_n, in order that the procedure can be applied;
(ii) the determination of the good classes of data f, g, Ω, and solutions u to which the procedure can be applied.

In this context it is also useful to recall that for every $P \in \Omega$ it holds

$$\frac{2\pi^{N/2}}{\Gamma(N/2)} u(P) = \int_\Gamma \left(u(x) \frac{\partial F(x,P)}{\partial \nu} - \frac{\partial u(x)}{\partial \nu} F(x,P) \right) d\Gamma - \int_\Omega f(x) F(x,P) dx. \qquad (4)$$

In order to study the previous problems, one has to study the fine properties of the simple and double layer potentials, appearing in (3) and (4). See for instance Fichera's paper [3], where many properties are studied, and in particular results of completeness are proved. (The modern potential theory studies the fine properties of the representation (4) for general Lipschitz domains and in a L_p framework.)

Magenes applied the method to the heat operator $E(u) = \Delta u - \frac{\partial u}{\partial t}$ in $\Omega \times (0,T)$, whose formal adjoint is $E^\star(u) = \Delta u + \frac{\partial u}{\partial t}$; the Green formula (2) becomes

$$\begin{aligned} &\int_0^T \int_\Omega E(u) w dx dt - \int_0^T \int_\Omega u E^\star(w) dx dt \\ &= \int_0^T \int_\Gamma \left(\frac{\partial u}{\partial \mathbf{n}} w - u \frac{\partial w}{\partial \mathbf{n}} \right) d\Gamma dt \\ &+ \int_\Omega u(T) w(T) dx - \int_\Omega u(0) w(0) dx. \end{aligned} \qquad (5)$$

Following the approach of Amerio and Fichera, Magenes used the fundamental solution of the heat equation, defined by $F(x,t;x',t') = \frac{1}{t'-t} \exp(-\frac{\|x'-x\|}{4(t'-t)})$ for $t' > t$ and $F(x,t;x',t') = 0$ for $t' \le t$. He also defined a class of solutions of the heat equation $E(u) = f$, assuming the boundary value in a suitable way and represented by potentials of simple and double layer.

These researches were followed [17] by the study of the so-called mixed problem, where the boundary is splitted in two parts: in the first one the boundary condition is of Dirichlet type, and in the other one the datum is the co-normal derivative. This problem was of particular difficulty for the presence of discontinuity in the data, even in the stationary case: see [16] (where the results are stated for $N = 2$, although they are valid for arbitrary N) and has stimulated many researches using potential theory not only of Magenes (see e.g. [18, 19]) but also of Fichera, Miranda, Stampacchia, ...

2 The Years in Genoa with G. Stampacchia

From the historical point of view, these researches show the change of perspective that occurred in Italy at that time in the study of these problems with the use of

- some first type of trace theorems (e.g. inspired by the results of Cimmino [2]);
- the introduction of the concept of weak solution;
- the use of general theorems of functional analysis (see e.g. [4]).

Under this point of view, the following summary of a conference of Magenes gives a typical account (see [20]). *Breve esposizione e raffronto dei più recenti sviluppi della teoria dei problemi al contorno misti per le equazioni alle derivate parziali lineari ellittiche del secondo ordine, soprattutto dal punto di vista di impostazioni "generalizzate" degli stessi* (*A short presentation and comparison of the most recent developments in the theory of mixed boundary value problems for second order elliptic linear partial differential equations, mainly from the point of view of "generalized" approaches to them*).

At the end of 1955 Magenes left the University of Modena for the University of Genoa, where he had G. Stampacchia as colleague. Stampacchia was a very good friend of Magenes from their years as students at Scuola Normale, since both where antifascist. Moreover, Magenes and Stampacchia were well aware of the fundamental change induced by the distribution theory and the Sobolev spaces in the calculus of variations and in the study of partial differential equations, particularly in the study of boundary value problems for elliptic equations (see for instance the bibliography of [20]).

They studied the works of L. Schwartz and its school, and specially the results on the mixed problem in the Hadamard sense. At the first *Réunion des mathématiciens d'expression latine*, in September 1957, Magenes and Stampacchia met J.-L. Lions. It was the beginning of a friendship, that would never stop. During the Spring 1958, J.-L. Lions gave at Genoa a series of talks on the mixed problems [5, 6], and in June 1958 Magenes and Stampacchia completed a long paper [22], that would have a fundamental influence on the Italian researches on elliptic partial differential equations. Indeed, that paper gives a general presentation of the results obtained up to that moment in France, United States, Sweden, Soviet Union by N. Aronszajn,