KELLOGG

FOUNDATIONS OF POTENTIAL THEORY

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BY

OLIVER DIMON KELLOGG
PROFESSOR OF MATHEMATICS IN HARVARD UNIVERSITY
CAMBRIDGE · MASSACHUSETTS · U.S.A.

, WITH 30 FIGURES

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Preface

The present volume gives a systematic treatment of potential functions. It takes its origin in two courses, one elementary and one advanced, which the author has given at intervals during the last ten years, and has a two-fold purpose: first, to serve as an introduction for students whose attainments in the Calculus include some knowledge of partial derivatives and multiple and line integrals; and secondly, to provide the reader with the fundamentals of the subject, so that he may proceed immediately to the applications, or to the periodical literature of the day.

It is inherent in the nature of the subject that physical intuition and illustration be appealed to freely, and this has been done. However, in order that the book may present sound ideals to the student, and also serve the mathematician, both for purposes of reference and as a basis for further developments, the proofs have been given by rigorous methods. This has led, at a number of points, to results either not found elsewhere, or not readily accessible. Thus, Chapter IV contains a proof for the general regular region of the divergence theorem (Gauss', or Green's theorem) on the reduction of volume to surface integrals. The treatment of the fundamental existence theorems in Chapter XI by means of integral equations meets squarely the difficulties incident to the discontinuity of the kernel, and the same chapter gives an account of the most recent developments with respect to the Dirichlet problem.

Exercises are introduced in the conviction that no mastery of a mathematical subject is possible without working with it. They are designed primarily to illustrate or extend the theory, although the desirability of requiring an occasional concrete numerical result has not been lost sight of.

Grateful acknowledgements are due to numerous friends on both sides of the Atlantic for their kind interest in the work. It is to my colleague Professor COOLIDGE that I owe the first suggestion to undertake it. To Professor OSGOOD I am indebted for constant encouragement and wise counsel at many points. For a careful reading of the manuscript and for helpful comment, I am grateful to Dr. ALEXANDER WEINSTEIN, of Breslau; and for substantial help with the proof, I wish to thank my pupil Mr. F. E. ULRICH. It is also a pleasure to acknowledge the generous attitude, the unfailing courtesy, and the ready coöperation of the publisher.

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O. D. Kellogg

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Chapter I.

The Force of Gravity.

1. The Subject Matter of Potential Theory.

While the theory of Newtonian potentials has various aspects, it is best introduced as a body of results on the properties of forces which are characterized by *Newtons Law of Universal Gravitation*[1]:

Every particle of matter in the universe attracts every other particle, with a force whose direction is that of the line joining the two, and whose magnitude is directly as the product of their masses, and inversely as the square of their distance from each other.

If, however, potential theory were restricted in its applications to problems in gravitation alone, it could not hold the important place which it does, not only in mathematical physics, but in pure mathematics as well. In the physical world, we meet with forces of the same character acting between electric charges, and between the poles of magnets.

But as we proceed, it will become evident that potential theory may also be regarded as the theory of a certain differential equation, known as LAPLACE'S. This differential equation characterizes the steady flow of heat in homogeneous media, it characterizes the steady flow of ideal fluids, of steady electric currents, and it occurs fundamentally in the study of the equilibrium of elastic solids.

The same differential equation in two dimensions is satisfied by the real and imaginary parts of analytic functions of a complex variable, and RIEMANN founded his theory of these functions on potential theory. Differential geometry, conformal mapping, with its applications to geographical maps, as well as other branches of mathematics, find important uses for Laplace's equation. Finally, the methods devised for the solution of problems of potential theory have been found to be of far wider applicability, and have exerted a profound influence on the theory of the differential equations of mathematical physics, both ordinary and partial, and on other branches of analysis[2].

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[1] *Philosophiae Naturalis Principia Mathematica*, Book III, Propositions I—VII. Formulated as above in THOMSON and TAIT, *Natural Philosophy*, Pt. II, p. 9.

[2] Indications on the literature will be found at the end of the book.

2. Newton's Law.

It is our experience that in order to set bodies in motion, or to stop or otherwise change their motion, we must exert forces. Accordingly, when we see changes in the motion of a body, we seek a cause of the character of a force. As bodies about us, when free to do so, fall toward the earth, we are accustomed to attribute to the earth an attracting power which we call the force of gravity. It is not at all obvious that the smaller bodies on the earth attract each other; if they do, the forces must be exceedingly minute. But we do see the effects of forces on the moon and planets, since they do not move in the straight lines we are accustomed to associate with undisturbed motion. To NEWTON it occurred that this deviation from straight line motion might be regarded as a continual falling, toward the earth in the case of the moon, and toward the sun in the case of the planets; this continual falling could then be explained as due to an attraction by the earth or sun, exactly like the attraction of the earth for bodies near it. His examination of the highly precise description of planetary motion which KEPLER had embodied in three empirical laws led, not only to the verification of this conjecture, but to the generalization stated at the beginning of the first section. The statement that all bodies attract each other according to this law has been abundantly verified, not only for heavenly bodies, but also for masses which are unequally distributed over the earth, like the equatorial bulge due to the ellipticity of the earth, and mountains, and finally for bodies small enough to be investigated in the laboratory.

The magnitude of the force between two particles, one of mass m_1, situated at a point P, and one of mass m_2, situated at Q, is given by Newton's law as

$$F = \gamma \frac{m_1 m_2}{r^2},$$

where r is the distance between P and Q. The constant of proportionality γ depends solely on the units used. These being given, its determination is purely a matter of measuring the force between two bodies of known mass at a known distance apart. Careful experiments have been made for this purpose, an account of which may be found in the *Encyclopedia Britannica* under the heading *Gravitation*[1]. If the unit of mass is the gramme, of length, the centimetre, of time, the second, and

[1] See also ZENNECK: Encyklopädie der Mathematischen Wissenschaften, Vol. V, pp. 25—67. Recently, measurements of a high degree of refinement have been made by Dr. P. R. HEYL, of the U. S. Bureau of Standards. See *A Redetermination of the Constant of Gravitation*, Proceedings of the National Academy of Sciences, Vol. 13 (1927), pp. 601—605.

The value of γ there given has been adopted here, although it should be noted that further experiments by Dr. HEYL are still in progress.

of force, the dyne, it is found that $\gamma = 6{\cdot}664 \times 10^{-8}$. If we borrow the result (p. 7) that a homogeneous sphere attracts as if concentrated at its center, we see that this means that two spheres of mass one gramme each, with centers one centimetre apart, will attract eachother with a force of ·00000006664 dynes.

In order to avoid this inconvenient value of γ, it is customary in potential theory to choose the unit of force so that $\gamma = 1$. This unit of force is called the *attraction unit*.

Exercises.

1. If the unit of mass is the pound, of length, the foot, of time, the second, and of force, the poundal, show that γ has the value $1{\cdot}070 \times 10^{-9}$. One foot contains 30·46 cm., and one pound, 453·6 gm.

2. Two homogeneous lead spheres, of diameter 1 ft. are placed in contact with each other. Compute the force with which they attract each other. A cubic foot of lead weights 710 pounds. Answer, about ·0000046 lb. This is approximately the weight of a square of medium weight bond paper, of side $^1/_4$ in.

3. Compute the mass of the earth, knowing the force with which it attracts a given mass on its surface, taking its radius to be 3955 miles. Hence show that the earth's mean density is about 5·5 times that of water. Newton inferred that the mean density lies between 5 and 6 times that of water.

4. Find the mass of the sun, it being given that the sun's attraction on the earth is approximately in equilibrium with the centrifugal force due to the earth's motion around the sun in a circle of $4{\cdot}90 \times 10^{11}$ feet. Answer, about 330,000 times the mass of the earth.

3. Interpretation of Newton's Law for Continuously Distributed Bodies.

Newton's law was stated in terms of particles. We usually have to deal, not with particles, but with continuously distributed matter. We then naturally think of dividing the body into small parts by the method of the integral calculus, adding the vector forces corresponding to the parts, and passing to the limit as the maximum chord of the parts approaches 0. This, in fact, is exactly what we shall do. But it should be pointed out that such a process involves an additional assumption. For no matter how fine the division, the parts are still not particles, Newton's law as stated is not applicable to them, and we have no means of determining the forces due to the parts.

The physical law which we shall adopt, and which may well be regarded simply as an amplified statement of Newton's law, is the following: *Given two bodies, let them be divided into elements after the manner of the integral calculus, and let the mass of each element be regarded as concentrated at some point of the element. Then the attraction which one body exerts on the other is the limit of the attraction which the corresponding system of particles exerts on the second system of particles, as the maximum chord of the elements approaches* 0. We shall revert to this assumption, and consider its legitimacy, on p. 22.

4. Forces Due to Special Bodies.

Because of their use in other problems of potential theory, because of the generalizations which they illustrate, and because of the practice which they give in dealing with Newtonian forces, the attractions due to special bodies are well worth study.

While each of two bodies attracts the other, the forces exerted are not equal vectors. Their magnitudes are equal, but they are oppositely directed. In order to avoid ambiguity it will be convenient to speak of one body as the attracting, and the other as the attracted body. This merely means that we are specifying the body the force on which we are determining. We shall also confine ourselves for the present to the case in which the attracted body is a unit particle. It will appear in § 11 (page 27) that the results are of wider significance than is at first evident. This section will be devoted to some illustrative examples.

Straight homogeneous segment. Let us consider a straight line segment, which we regard as having mass, so distributed that the mass on any interval is proportional to the length of the interval. The constant factor of proportionality λ is called the *linear density*. We have here an idealization of a straight wire, which is a better approximation the smaller the diameter of the wire relatively to its length and the distance away of the attracted particle.

Let axes be chosen so that the ends of the wire are the points $(0, 0, 0)$ and $(l, 0, 0)$. As a first case, let the attracted particle be in line with the wire, at $(x, 0, 0)$, $x > l$. Let the wire be divided into intervals by the points $\xi_0 = 0$, ξ_1, ξ_2, $\ldots \xi_n = l$ (fig. 1). Then the interval (ξ_k, ξ_{k+1}) carries a mass $\lambda \Delta \xi_k$, which, by our physical law, is to be regarded as concentrated at some point ξ_k' of the interval. The force due to the particle thus constructed will lie along the x-axis, and will be given, in attraction units, by

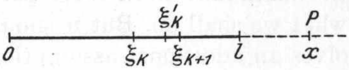

Fig. 1.

$$\Delta X_k = -\frac{\lambda \Delta \xi_k}{|x - \xi_k'|^2},$$

$$\Delta Y_k = 0, \quad \Delta Z_k = 0.$$

The force due to the whole segment will be the limit of the sum of the forces due to the system of particles, or

$$X = -\int_0^l \frac{\lambda\, d\xi}{(x-\xi)^2}, \quad Y = 0, \quad Z = 0,$$

or

$$X = -\frac{\lambda l}{x(x-l)}, \quad Y = 0, \quad Z = 0.$$

The result may be given a more suggestive form by introducing the total mass $M = \lambda l$, and considering at what point of the segment a particle of that mass should be placed in order to yield the same attraction on a unit particle at $P(x, 0, 0)$. If c is the coördinate of this point,

$$X = -\frac{\lambda l}{x(x-l)} = -\frac{M}{c^2} \quad \text{and} \quad c = \sqrt{x(l-x)}\,.$$

Thus the wire attracts a unit particle at P as if the mass of the wire were concentrated at a point of the wire whose distance from P is the geometric mean of the distances from P of the ends of the wire.

As P approaches the nearer end of the wire, the force becomes infinite, but only like the inverse first power of the distance of P from this end, although a particle would produce a force which became infinite like the inverse square of the distance. The difference is that in the case of the particle, P draws near to the whole mass, whereas in the case of the wire the mass is distributed over a segment to only one of whose points does P draw arbitrarily near.

As P recedes farther and farther away, the *equivalent particle* (as we shall call the particle with the same mass as the wire, and with the same attraction on a unit particle at P) moves toward the mid-point of the wire, and the attraction of the wire becomes more and more nearly that of a fixed particle at its mid-point. An examination of such characteristics of the attraction frequently gives a satisfactory check on the computation of the force.

Let us now consider a second position of the attracted particle, namely a point $P\left(\frac{l}{2}, y, 0\right)$ on the perpendicular bisector of the material segment (fig. 2). The distance r of the attracted particle from a point $(\xi_k', 0, 0)$ of the interval (ξ_k, ξ_{k+1}) is given by

$$r^2 = \left(\xi_k' - \frac{l}{2}\right)^2 + y^2,$$

and the magnitude of the force at P, due to a particle at this point, whose mass is that on the interval (ξ_k, ξ_{k+1}) is

$$\Delta F_k = \frac{\lambda \Delta \xi_k}{\left(\xi_k' - \frac{l}{2}\right)^2 + y^2}\,.$$

This force has the direction cosines

$$\frac{\xi_k' - \frac{l}{2}}{r}, \quad \frac{-y}{r}, \quad 0,$$

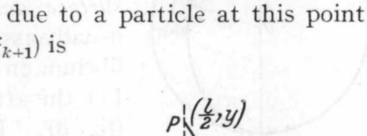

Fig. 2.

and therefore the components

$$\Delta X_k = \frac{\lambda\left(\xi_k' - \frac{l}{2}\right)\Delta\xi_k}{\left[\left(\xi_k' - \frac{l}{2}\right)^2 + y^2\right]^{\frac{3}{2}}}, \quad \Delta Y_k = \frac{-\lambda y \Delta \xi_k}{\left[\left(\xi_k' - \frac{l}{2}\right)^2 + y^2\right]^{\frac{3}{2}}}, \quad \Delta Z_k = 0\,.$$

The limits of the sums of these components give the components of the attraction of the segment

$$X = \lambda \int_0^l \frac{\left(\xi - \frac{l}{2}\right) d\xi}{\left[\left(\xi - \frac{l}{2}\right)^2 + y^2\right]^{\frac{3}{2}}}, \quad Y = -\, y\lambda \int_0^l \frac{d\xi}{\left[\left(\xi - \frac{l}{2}\right)^2 + y^2\right]^{\frac{3}{2}}}, \quad Z = 0.$$

The first integral vanishes, since the integrand has equal and opposite values at points equidistant from $\xi = \frac{l}{2}$. The second integral is easily evaluated, and gives

$$Y = -\frac{\lambda l}{y\sqrt{\left(\frac{l}{2}\right)^2 + y^2}} = -\frac{M}{c^2},$$

if c is the geometric mean of the distances from P of the nearest and farthest points of the wire. The equivalent particle, is thus seen to lie *beyond* the wire as viewed from P. This fact is significant, as it shows that there does not always exist *in* a body a point at which its mass can be concentrated without altering its attraction for a second body. Our physical law does not assert that such a point exists, but only that if one be assumed in each of the parts into which a body is divided, the errors thereby introduced vanish as the maximum chord of the parts approaches 0.

Spherical shell. Let us take as a second illustration the surface of a sphere with center at O and radius a, regarding it as spread with mass such that the mass on any part of the surface is proportional to the area of that part. The constant factor of proportionality σ is called the *surface density*. We have here the situation usually assumed for a charge of electricity in equilibrium on the surface of a spherical conductor[1]. Let the attracted particle be at $P(0, 0, z)$, $z \neq a$ (fig. 3). Let ΔS_k denote a typical element of the surface, containing a point Q_k with spherical coördinates $(a, \varphi'_k, \vartheta'_k)$. Then the magnitude of the element of the force at P due to the mass $\sigma \Delta S_k$ of the element of surface ΔS_k, regarded as concentrated at Q_k is

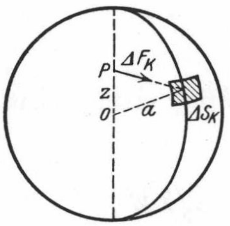

Fig. 3.

$$\Delta F_k = \frac{\sigma \Delta S_k}{r_k^2} = \frac{\sigma \Delta S_k}{a^2 + z^2 - 2az\cos\vartheta'_k}.$$

By symmetry, the force due to the spherical shell will have no component perpendicular to the z-axis, so that we may confine ourselves

[1] See Chapter VII (page 176).

to the components of the elements of force in the direction of the z-axis. The cosine of the angle between the element of force and this axis is

$$\frac{a \cos \vartheta_k' - z}{r_k},$$

so that

$$\Delta Z_k = \frac{\sigma (a \cos \vartheta_k' - z) \Delta S_k}{[a^2 + z^2 - 2 a z \cos \vartheta_k']^{\frac{3}{2}}},$$

and the total force is given by the double integral over the surface of the sphere

$$Z = \sigma \iint\limits_S \frac{(a \cos \vartheta - z) \, dS}{[a^2 + z^2 - 2 a z \cos \vartheta]^{\frac{3}{2}}}.$$

This is equivalent to the iterated integral

$$Z = \sigma a^2 \int_0^{\pi} \int_0^{2\pi} \frac{(a \cos \vartheta - z) \, d\varphi \sin \vartheta \, d\vartheta}{[a^2 + z^2 - 2 a z \cos \vartheta]^{\frac{3}{2}}},$$

$$= 2 \pi \sigma a^2 \int_0^{\pi} \frac{(a \cos \vartheta - z) \sin \vartheta \, d\vartheta}{[a^2 + z^2 - 2 a z \cos \vartheta]^{\frac{3}{2}}}.$$

In evaluating this last integral (which may be done by introducing r as the variable of integration), it must be kept in mind that

$$r = \sqrt{a^2 + z^2 - 2 a z \cos \vartheta}$$

is a distance, and so essentially positive. Thus, its value for $\vartheta = 0$ is $|a - z|$, that is $a - z$ or $z - a$ according as $a > z$ or $z > a$. The result is

$$Z = -\frac{4 \pi a^2 \sigma}{z^2} = -\frac{M}{z^2} \quad \text{for} \quad z > a,$$

$$Z = 0 \qquad\qquad\qquad \text{for} \quad 0 \leqq z < a.$$

That is, *a homogeneous spherical shell attracts a particle at an exterior point as if the mass of the shell were concentrated at its center, and exercises no force on a particle in its interior.*

Homogeneous solid sphere. If a homogeneous solid sphere be thought of as made up of concentric spherical shells, it is a plausible inference that the whole attracts a particle as if the sphere were concentrated at its center. That this is so, we verify by setting up the integral for the attraction. Let $\varkappa$ denote the constant ratio of the mass of any part of the sphere to the volume of the part, that is, the *density*. The mass $\varkappa \Delta V$ in the element ΔV, regarded as concentrated at the point

$Q(\varrho, \varphi, \vartheta)$ will exert on a unit particle at $P(z, 0, 0)$, a force whose magnitude is

$$\Delta F = \frac{\varkappa \Delta V}{\varrho^2 + z^2 - 2\varrho z \cos\vartheta}$$

and whose component in the direction of the z-axis is therefore

$$\Delta Z = \frac{\varkappa(\varrho\cos\vartheta - z)\Delta V}{[\varrho^2 + z^2 - 2\varrho z\cos\vartheta]^{\frac{3}{2}}}.$$

Hence, for the total force,

$$Z = \varkappa \int_0^a \int_0^\pi \int_0^{2\pi} \frac{(\varrho\cos\vartheta - z)\, d\varphi \sin\vartheta\, d\vartheta\, \varrho^2\, d\varrho}{[\varrho^2 + z^2 - 2\varrho z\cos\vartheta]^{\frac{3}{2}}}.$$

The two inner integrals have been evaluated in the previous example. We have only to replace a by ϱ and evaluate the integral with respect to ϱ. The result is

$$Z = -\frac{4\pi\varkappa}{z^2}\int_0^a \varrho^2\, d\varrho = -\frac{4\pi\varkappa a^3}{3z^2} = -\frac{M}{z^2},$$

as was anticipated.

Further examples will be left as exercises to the reader in the following sections. We take them up in the order of multiplicity of the integrals expressing the components of the force.

5. Material Curves, or Wires.

We take up first the case in which the attracting body is a material curve. Consider a wire, of circular cross-section, the centers of the circles lying on a smooth curve C. If we think of the mass between any pair of planes perpendicular to C as concentrated on C between these planes, we have the concept of a *material curve*. By the *linear density* λ of the material curve, or where misunderstanding is precluded, by the density, at a point Q, we mean the limit of the ratio of the mass of a segment containing Q to the length of the segment, as this length approaches 0.

Our problem is now to formulate the integrals giving the force exerted by a material curve C on a particle at P. Let the density of C be given as a function λ of the length of arc s of C measured from one end. We assume that λ is continuous. Let C be divided in the usual way into pieces by the points $s_0 = 0, s_1, s_2, \ldots, s_n = l$, and let us consider the attraction of a typical piece Δs_k. The mass of this piece will lie between the products of the least and greatest value of λ on the piece by the length of the piece, and therefore it will be equal to $\lambda_k' \Delta s_k$, where λ_k' is a properly chosen mean value of λ. A particle with this mass, situated at a point Q_k of the piece, will exert on a unit particle at $P(x, y, z)$ a force whose magnitude is

$$\Delta F = \frac{\lambda_k' \Delta s_k}{r_k^2}, \quad r_k = \overline{PQ_k}.$$

If ξ_k, η_k, ζ_k are the coördinates of Q_k, the direction cosines of this force are

$$\cos\alpha = \frac{\xi_k - x}{r_k}, \quad \cos\beta = \frac{\eta_k - y}{r_k}, \quad \cos\gamma = \frac{\zeta_k - z}{r_k},$$

so that the components of the force due to the typical piece are

$$\Delta X_k = \frac{\lambda_k' (\xi_k - x) \Delta s_k}{r_k^3}, \quad \Delta Y_k = \frac{\lambda_k' (\eta_k - y) \Delta s_k}{r_k^3}, \quad \Delta Z_k = \frac{\lambda_k' (\zeta_k - z) \Delta s_k}{r_k^3}.$$

The components in each of the three directions of the axes corresponding to all the pieces of the wire are now to be added, and the limits taken as the lengths of the pieces approach 0. The results will be the the components of the force on the unit particle at P due to the curve:

$$X = \int_C \frac{\lambda(\xi - x)}{r^3}\, ds,$$

(1) $$Y = \int_C \frac{\lambda(\eta - y)}{r^3}\, ds,$$

$$Z = \int_C \frac{\lambda(\zeta - z)}{r^3}\, ds.$$

We shall sometimes speak of a material curve as a *wire*. We shall also speak of the attraction on a unit particle at P simply as the *attraction at P*. An illustration of the attraction of a wire was given in the last section. Further examples are found in the following exercises, which should be worked and accompanied by figures.

Exercises.

1. Find the attraction of a wire of constant density having the form of an arc of a circle, at the center of the circle. Show that the equivalent particle is distant $\sqrt{\frac{\alpha}{\sin\alpha}}$ from the center, where a is the radius of the arc and 2α is the angle it subtends at the center. The equivalent particle is thus not in the body. But there is a point on the wire such that if the total mass were concentrated there, the *component* of its attraction along the line of symmetry of the arc would be the actual attraction. Find this point.

2. Find the attraction of a straight homogeneous piece of wire, at any point P of space, not on the wire. Show that the equivalent particle lies on the bisector of the angle APB, A and B being the ends of the wire, and that its distance c from P is the geometric mean of the two quantities: the length of the bisector between P and the wire, and the arithmetic mean of the distances $\overline{PA}$ and $\overline{PB}$.

3. Show, by comparing the attraction of corresponding elements, that a straight homogeneous wire exercises the same force at P as a tangent circular wire with center at P, terminated by the same rays from P, and having the same linear density as the straight wire.

4. Find the attraction of a homogeneous circular wire at a point P on the axis of the wire. Show that the distance c of the equivalent particle is given by $c = d\sqrt{\frac{d}{d'}}$, where d is the distance of P from the wire, and d' its distance from the plane of the wire.

5. In Exercise 2, show that if the wire be indefinitely lengthened in both directions, the force approaches a limit in direction and magnitude (by definition, the force due to the infinite wire), that this limiting force is perpendicular to the wire, toward it, and of magnitude $\frac{2\lambda}{r}$, where λ is the linear density of the wire, and r the distance of P from it.

6. Material Surfaces, or Laminas.

Consider a thin metallic plate, or shell, whose faces may be thought of as the loci formed by measuring off equal constant distances to either side of a smooth surface S on the normals to S. We arrive at the notion of a *material surface* or *lamina* by imagining the mass of the shell concentrated on S in the following way: given any simple closed curve on S, we draw the normals to S through this curve; the mass included within the surface generated by these normals we regard as belonging to the portion of S within the curve, and this for every such curve. The *surface density*, or if misunderstanding is precluded, the density, of the lamina at Q is defined as the limit of the ratio of the mass of a piece of S containing Q to the area of the piece, as the maximum chord of the piece approaches 0. In addition to the terms material surface and lamina, the expressions *surface distribution*, and *surface spread*, are used.

As we have noted in studying the attraction of a material spherical surface, the notion of surface distribution is particularly useful in electrostatics, for a charge in equilibrium on a conductor distributes itself over the surface.

Now, according to Couloumb's law, two point charges of electricity in the same homogeneous medium, exert forces on each other which are given by Newton's law with the word mass replaced by charge, except that if the charges have like signs, they repel each other, and if opposite signs, they attract each other. A constant of proportionality will be determined by the units used and by the medium in which the charges are situated. Because of the mathematical identity, except for sign, between the laws governing gravitational and electric forces, any problem in attraction may be interpreted either in terms of gravitation or in terms of electrostatics. Thus, in the case of an electrostatic charge