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IXth International Congress on Mathematical Physics

Cynhaliwyd IXfed Gyngres Ryngwladol IAMP yng Ngholeg y Brifysgol Abertawe o'r 17eg hyd at y 27ain o Orffennaf 1988. Cynt cynhaliwyd y Gyngres hon ym Moscow, Warsaw, Kyoto,Rhufain, Lausanne, Berlin, Boulder a Marseille. Felly hon oedd y gyngres gyntaf i'w chynnal ym Mhrydain a'r cyntaf yng Nghymru. Bu'r Sector Mathemateg, Coleg y Brifysgol Abertawe yn westeiwyr i'r Gyngres, ac fe ddymuna'r trefnwyr lleol I M Davies a A Truman ddiolch i Barry Simon, Michael Atiyah, Sergei Novikov a John Lewis oherwydd heb eu cymorth ni fyddai wedi bod yn bosib cynnal y Gyngres. Yn ogystal, diolchiadau arbennig i Betty Williams ac i aelodau'r Sector Mathemateg, Yr Adran Fathemateg a Chyfrifiadureg Coleg y Brifysgol Abertawe. Cafwyd boddhâd neilltuol wrth groesawu cynifer o gynrychiolwyr o Rwsia yn ystod y cyfnod yma o Berestroica.

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Preface

This volume contains the proceedings of the IXth International Congress of the International Association of Mathematical Physics. It was held in Swansea, United Kingdom, between the dates of 17th July and 27th July 1988 at University College of Swansea. There were over 300 participants from more than 20 different countries. It was especially pleasing to have 24 participants from the USSR. This congress followed a similar pattern to the previous eight congresses.

The congress had the support of the Science and Engineering Research Council, the International Union of Pure and Applied Physics, the International Mathematical Union, the London Mathematical Society, the International Association of Mathematical Physics and the Institute of Physics. The congress would not have been as successful as it was had it not been for the comprehensive social programme which had the support of the City of Swansea, Digital Equipment Corporation, British Petroleum, the Midland Bank, the Welsh Tourist Board, Lloyds Bank, Deloitte, Haskins and Sells and Peter Llewellyn (Konica). We thank all the organisations mentioned herein for their generous support.

The congress consisted of plenary lectures, special sessions and poster sessions. The plenary lectures were intended to be accessible to all participants and the plenary speakers were invited by the scientific organising committee to give reviews of their own field of interest. The special sessions comprised of approximately six lectures of a technical nature, the session organisers being responsible for their particular programme and inviting speakers. The session organisers were themselves invited by the scientific organising committee to act in this capacity. The poster session was split over three afternoons with reference to the timing of the special sessions.

The congress was opened by Barry Simon and the participants were welcomed by David Evans.

We would like to thank all those who helped to make the congress the great success it was. Special mention must go to the members of the Mathematics Division of the Department of Mathematics and Computer Science who provided the personnel to manage the day to day running of the congress. Special thanks must go to Sir Michael Atiyah, Sergei Novikov and John Lewis without whose help the Congress could not have taken place.

B Simon

A Truman

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February 1989

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Conformal symmetry and its extensions

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ABSTRACT: The theory of the unitary representations of the algebra of the conformal group in two dimensions (the Virasoro algebra) is reviewed, in the context of applications to statistical physics and conformal field theory. The role of modular invariance and the construction and classification of modular invariants is discussed. The desirability of finite reducibility of modular invariant representations leads to the consideration of extensions of conformal symmetry. The coset construction provides a unifying framework within which these various concepts can be elucidated.

1. INTRODUCTION

The main subject of this review is what has been learnt in the last few years about the group of conformal (*i.e.* angle-preserving) transformations in two dimensions. The special feature of two dimensions is that the conformal group is infinite-dimensional, whereas it is not in higher dimensions. This infinite-dimensional group is relevant in (essentially) two-dimensional situations where the interactions are local and scale invariant, because, in a local theory, scale invariance implies conformal invariance. Hence, it is important in analysing the critical behaviour of suitable two-dimensional statistical systems (*e.g.* systems of interacting spin variables), and in string theories because of the two-dimensional nature of the string world-sheet.

The algebra of the group of conformal transformations in two dimensions consists of two commuting copies of the *Virasoro algebra* (Virasoro 1970), $\hat{v}$. It has a basis consisting of L_n, $n \in \mathbf{Z}$, and a central element, c, satisfying the commutation relations,

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}m(m^2-1)\delta_{m,-n}, \tag{1a}$$

$$[L_n, c] = 0. \tag{1b}$$

In many mathematical and physical contexts, the representations of $\hat{v}$ which are relevant satisfy two conditions: they are *unitary*, i.e. they satisfy the hermiticity condition

$$L_n^\dagger = L_{-n}, \tag{2}$$

with respect to a positive definite scalar product; and they have the *"positive energy"* property that L_0 is bounded below. In an irreducible unitary representation the central element c takes a fixed real value.

The topics will be covered in this review are as follows. In Section 2, we shall discuss the ***Representation Theory*** of the Virasoro algebra. We shall explain how its unitary representations can be constructed in terms of the unitary representations of affine Kac-Moody algebras, especially $\widehat{su}(2)$, using the ***coset construction*** (Goddard, Kent and Olive 1985, 1986). [For a longer review of these topics, see e.g. Goddard and Olive 1986] Next, in Section 3, we shall discuss ***Characters and Modular Invariance***, and, in particular, the covariance of characters of the Virasoro algebra under the modular group $SL(2, \mathbf{Z})$; this is relevant both in string theory and in two-dimensional statistical mechanics. The power of conformal invariance in two-dimensional quantum field theory was made evident by the work of Polyakov (1970) and the seminal paper of Belavin, Polyakov and Zamolodchikov (1984). The constraints following from modular invariance have been emphasised and analysed by Cardy (1986), by Gepner and Witten (1986) and by Cappelli, Itzykson and Zuber (1987a,b). Some years earlier, Nahm (1976) had stressed their role in string theory.

Finally, in Section 4, we consider ***Extensions of Conformal Symmetry***. The simplest sort of extension is to include supersymmetry, to obtain superconformal symmetry. But once one generalises from a Lie algebra to a superalgebra there is no reason not to consider more general sorts of algebra, following Zamolodchikov and Fateev (1985), defined by operator product expansions, such as the parafermionic algebras. These can be used to characterise Conformal Field Theories. They enable us to construct a wide class of theories which are not only modular invariant but also finitely-reducible under the symmetry algebra considered.

The theme of our presentation will be that these various aspects of conformal symmetry (representation theory, modular invariance, etc.) can all be understood from a unified point of view using the coset construction. For example, it seems that the known examples of extended conformal algebras can be described in terms of this construction and it possible that all "rational conformal field theories" are realisable as coset theories (Kastor *et al.* 1988), though it is not clear why this construction should be so universal.

2. REPRESENTATION THEORY

2.1 Conditions for unitarity

We consider positive energy representations of the Virasoro algebra, *i.e.* those for which L_0 is diagonalisable and its spectrum is bounded below. This is a typical physical requirement because L_0 often corresponds to an energy or dilatation operator, or something similar. Then, since

$$L_0 L_n = L_n(L_0 - n), \tag{3}$$

the L_n, $n > 0$, act as lowering operators for the eigenvalues of L_0. Thus, if $|h\rangle$ is an eigenvector of L_0 corresponding to the lowest eigenvalue h,

$$L_0|h\rangle = h|h\rangle, \tag{4a}$$
$$L_n|h\rangle = 0, \qquad n > 0. \tag{4b}$$

Such a state $|h\rangle$ is usually called a ***highest weight state***. It is not difficult to see that the whole of an irreducible positive energy representation is built up from such a highest weight state by the action of the algebra, and that the representation space is spanned by states of the form

$$(L_{-1})^{n_1}(L_{-2})^{n_2}\dots(L_{-r})^{n_r}|h\rangle. \tag{5}$$

Such a representation is called an irreducible ***highest weight representation*** and it is characterised by the pair of numbers (c, h). Any unitary positive energy representation is the direct sum of highest weight representations.

If the representation possesses a scalar product with respect to which the hermiticity condition (2) holds, the scalar product of any two states of the form (5) can be calculated in terms of c and h, using the commutation relations (1). Thus the question arises of for which values of (c, h) is the scalar product positive definite. This is of considerable importance because in some applications only representations which are *unitary* in this sense can occur. To analyse this question, we consider the matrix, $\mathcal{M}_N(c, h)$, formed by the scalar products of the states (5) for which L_0 has eigenvalue $h + N$, i.e. those for which $\sum jn_j = N$. This is a $\pi(N) \times \pi(N)$ dimensional matrix, where

$$\sum_{N=0}^{\infty} \pi(N)q^N \equiv \prod_{n=1}^{\infty}(1-q^n)^{-1}, \tag{6}$$

i.e. $\pi(N)$ is the number of partitions of N. In order for the representation to be unitary, these matrices need to be positive semi-definite for each positive N. The first step towards deciding when this happens was the remarkable formula for $\det \mathcal{M}_N(c, h)$ which was given by Kac (1979) and proved by Feigin and Fuks (1982). This takes the form

$$\det \mathcal{M}_N(c,h) = A_N \prod_{1\le p,q\le N} (h - h_{p,q}(c))^{\pi(N-pq)}, \tag{7}$$

where A_N is independent of h and c, the product is over positive integers p and q with $pq \le N$ and

$$h_{p,q}(c) = \frac{1}{24}(c-1) + \left[\tfrac{1}{2}p\beta_+ + \tfrac{1}{2}q\beta_-\right]^2, \tag{8a}$$

with

$$\beta_\pm = \frac{\sqrt{1-c} \pm \sqrt{25-c}}{\sqrt{24}}. \tag{8b}$$

This formula enabled Friedan, Qiu and Shenker (1984, 1986) [FQS] to find necessary conditions on (c, h) for unitarity; later we shall see, by explicit construction, that these conditions are also sufficient.

To see something of the way the conditions on (c, h) arise, consider

$$\begin{aligned}\|L_{-n}|h\rangle\|^2 &= \langle h|[L_n, L_{-n}]|h\rangle, \qquad \text{if } n > 0,\\ &= \{2nh + \frac{c}{12}n(n^2-1)\}\||h\rangle\|^2,\end{aligned} \tag{9}$$

which must be non-negative for all positive n. From this it immediately follows that

$$c \ge 0, \qquad h \ge 0. \tag{10}$$

It is not difficult to establish using Kac's formula that all the values

$$c \ge 1, \qquad h \ge 0, \tag{11}$$

correspond to unitary representations. The point $(c, h) = (0,0)$ corresponds to the trivial representation, i.e. $L_n \equiv 0$. It is tempting to suppose that this is the only unitary representation apart from the continuum of representations (11), but actually there are three representations well-known from the spinning string theory of Ramond-Neveu-Schwarz (Ramond 1971; Neveu and Schwarz 1971; Neveu, Schwarz and Thorn 1971; see also Bardacki and Halpern 1971), corresponding to a single periodic (Ramond) or anti-periodic (Neveu-Schwarz) fermion field. These correspond to $c = \frac{1}{2}$ and $h = 0$ or $\frac{1}{2}$ (NS case) and $h = \frac{1}{16}$ (R case). In fact, $c = 0$ and $\frac{1}{2}$ are the first two terms in a discrete series of unitary representations.

The result established by FQS is that for a unitary highest weight representation it is necessary that *either*

$$c \geq 1, \qquad h \geq 0, \tag{12a}$$

or

$$c = 1 - \frac{6}{(m+2)(m+3)}, \qquad m = 0, 1, \ldots, \tag{12b}$$

$$h = \frac{[(m+3)p - (m+2)q]^2 - 1}{4(m+2)(m+3)}, \tag{12c}$$

where $p = 1, 2, \ldots m+1$ and $q = 1, 2, \ldots p$. [The values (12c) of h are obtained by substituting the values c of (12b) into (8).] It turns out that the conditions (12) are also sufficient for unitarity (Goddard, Kent and Olive 1985, 1986).

2.2 Connections with statistical physics

It is perhaps helpful at this point to give some physical interpretation to the quantities c and h by explaining something of their significance in the statistical physics of two-dimensional systems. At a critical point of a suitable two-dimensional statistical system, scaling behaviour occurs which can be described by a two-dimensional conformally invariant Euclidean quantum field theory, a *conformal field theory*. The measurable critical exponents in such a theory are linear combinations of the scaling dimensions of appropriate fields. The conformal symmetry means that the states of the field theory form representations of the Lie algebra of the conformal group (or more accurately a central extension of it). In two dimensions this algebra consists of two commuting copies of the Virasoro algebra. For systems possessing the "reflection positivity" property these representations have to be unitary in the sense described above. Many systems of interest have this property and we shall restrict attention to them. The value of the central element c is common to both copies of the algebra and is a characteristic of the theory under consideration. The scaling dimensions of fields are related to the values of the highest weights h. Thus, the list of unitary representations places important restrictions on the possible scaling dimensions in the theory. In the last couple of years it has been realised that the possible combinations of irreducible representations that can occur in a theory are further restricted in suitable physical contexts by the requirement of modular invariance.

Using the complex plane to parameterize two-dimensional Euclidean space, a basis for the infinitesimal conformal transformations is provided by the transformations

$$z \mapsto z + \varepsilon z^{n+1}, \tag{13}$$

with ε a complex number. The corresponding generator can be denoted by $\varepsilon L_n + \varepsilon^* \bar{L}_n$. Here ε^* is the complex conjugate of ε but $\bar{L}_n$ is independent of L_n. In a conformally