Multivariable

CALCULUS

from Graphical, Numerical, and Symbolic Points of View

Preliminary Edition

OSTEBEE • ZORN

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About this preliminary edition: notes for instructors

This book aims to do what its title suggests: present multivariable calculus from graphical, numerical, and symbolic points of view. In doing so, we continue the philosophy and viewpoints embodied in our two volumes on single-variable calculus, *Calculus from Graphical, Numerical, and Symbolic Points of View*, also published by Saunders College Publishing. For many more details on philosophy, strategy, use of technology, and other issues, see either those volumes or our World Wide Web site: `http://www.stolaf.edu/people/zorn/ozcalc`

Various views. We aim to focus on the main *concepts* of multivariable calculus: derivative and integral in their higher-dimensional versions, linear approximation, parametrization, vector fields and vector operations, the multivariable analogues of the fundamental theorem of calculus, and a few geometric and physical applications. As in our treatment of single-variable calculus, the key strategy for improving conceptual understanding is to combine, compare, and move among graphical, numerical, and algebraic viewpoints.

Audience and prerequisites. The text addresses a general mathematical audience: mathematics majors, science and engineering majors, and non-science majors. We assume a little more mathematical maturity than for single-variable calculus, but the presentation is not rigorous in the sense of mathematical analysis. We want students to encounter, understand, and use the main concepts and methods of multivariable calculus, and to see how they extend the simpler objects and ideas of elementary calculus. A fully rigorous logical development belongs later in a student's mathematical education.

We assume that students have the "usual" one-year, single-variable calculus preparation, but little or nothing more than that. A basic familiarity with numerical integration techniques (such as the midpoint and trapezoid rules) is helpful, but it could be developed enroute if necessary. (We do not assume that students have studied our own texts!)

Although we stress linear functions and linear approximation, we do not assume that students have formal experience with linear algebra. Vectors are used throughout, but are introduced "from scratch." Matrices are used only occasionally, mainly in the last two chapters. Students for whom matrices are entirely new may need a little informal help with such rudiments as matrix multiplication. A few determinants appear in the last two chapters, but only for 2×2 and 3×3 matrices.

Technology. Technology is an important tool for illustrating and comparing graphical, numerical, and symbolic viewpoints in calculus—especially in multivariable calculus, where calculations can be messy, and where geometric intuition is harder to come by. Although we refer often to *Maple*, other programs (*Mathematica*, *Derive*, the TI-92, etc.) would do just as well; "translating" *Maple* references to other platforms should cause little difficulty. However, we strongly recommend that students have access to *some* capable and flexible technology, especially for graphical representations.

Exercises and solutions. Exercises in multivariable calculus are often somewhat more involved, involve more steps, and may be more open-ended than those in single-variable calcuclus. With this in mind, the *Solutions Manual* provides relatively complete solutions, rather than just answers, to many of the exercises. Some of the exercises and solutions could be used to augment and extend the formal Examples in the text.

Advice from you. We appreciate hearing instructors' comments, suggestions, and advice on this preliminary edition. Suggestions received from users of preliminary versions of our single-variable texts helped us revise them in later editions. Our physical and e-mail addresses are below.

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How to use this book: notes for students

All authors want their books to be *used*: read, studied, thought about, puzzled over, reread, underlined, disputed, understood, and, ultimately, enjoyed. So do we.

That might go without saying for *some* books—beach novels, user manuals, field guides, etc.—but it may need repeating for a calculus textbook. We know as teachers (and remember as students) that mathematics textbooks are too often read *backwards*: faced with Exercise 231(b) on page 1638, we've all shuffled backwards through the pages in search of something similar. (Very often, moreover, our searches were rewarded.)

A textbook isn't a novel. It's a peculiar hybrid of encyclopedia, dictionary, atlas, anthology, daily newspaper, shop manual, *and* novel—not exactly light reading, but essential reading nevertheless. Ideally, a calculus book should be read in *all* directions: left to right, top to bottom, back to front, and even front to back. That's a tall order. Here are some suggestions for coping with it.

Read the narrative. Each section's narrative is designed to be read from beginning to end. The examples, in particular, are supposed to illustrate ideas and make them concrete—not just serve as templates for homework exercises.

Read the examples. Examples are, if anything, more important than theorems, remarks, and other "talk." We use examples both to show already-familiar calculus ideas "in action," and to set the stage for new ideas.

Read the pictures. We're serious about the "graphical points of view" mentioned in our title. The pictures in this book are not "illustrations" or "decorations." The are an important part of the language of calculus. An ability to think "pictorially"—as well as symbolically and numerically—about mathematical ideas may be the most important benefit calculus can offer.

Read the language. Mathematics is not a "natural language" like English or French, but it has its own vocabulary and usage rules. Calculus, especially, relies on careful use of technical language. Words like **rate**, **amount**, **concave**, **stationary point**, and **root** have precise, agreed-upon mathematical meanings. Understanding such words goes a long way toward understanding the mathematics they convey; misunderstanding the words leads inevitably to confusion. Whenever in doubt, consult the index.

Read the instructors' preface (if you like). Get a jump on your teacher.

In short: *read the book.*

A last note

Why study calculus at all? There are plenty of good practical and "educational" reasons: because it's good for applications; because higher mathematics requires it; because it's good mental training; because other majors require it; because jobs require it. Here's another reason to study calculus: because calculus is among our species' deepest, richest, farthest-reaching, and most beautiful intellectual achievements. We hope this book will help you see it in that spirit.

A last request

Last, a request. We sincerely appreciate—and take very seriously—students' opinions, suggestions, and advice on this book. We invite you to offer your advice, either through your teacher or by writing us directly. Our addresses appear below.

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June, 1996

Contents

Chapter 1

Multivariable Calculus: A First Look

1.1 Three-dimensional space

Single-variable calculus is done mainly in the two-dimensional xy-plane. The Euclidean plane, also known as $\mathbb{R}^2$, is the natural home of such familiar calculus objects as the graph $y = f(x)$, tangent lines to this graph at various points, and the various regions whose areas we might measure by integration.

To do *multivariable* calculus we'll need more room. The graph of $z = f(x, y)$, where f is a function of two input variables, lives in *three*-dimensional xyz-space. With three dimensions to work in we'll be able to "see" not only these graphs but also a variety of multivariable analogues of derivatives and integrals. This section explores three-dimensional Euclidean space, also known as $\mathbb{R}^3$, where we'll be spending most of our time.

In another sense, of course, we spend *all* of our time in $\mathbb{R}^3$. In everyday use, the word "space" connotes three physical dimensions. The intuition we gain from living in three spatial dimensions is often useful in mentally picturing and manipulating the objects of multivariable calculus. Familiar as it is, however, three-dimensional space poses special problems for visualization. Two-dimensional pictures (on paper or on a computer screen) of three-dimensional objects are *always* more or less distorted or incomplete. Minimizing such problems is an active science (and an art) in its own right; doing so means carefully controlling viewpoint, perspective, shading, lighting, and other factors.

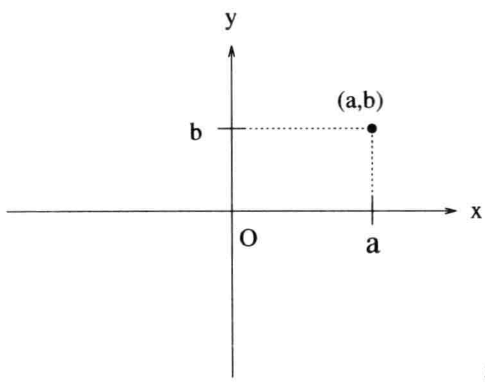
This book's main subject is multivariable calculus, not computer graphics➻ or technical drawing, so we'll draw pictures to illustrate ideas as simply as possible—not necessarily to look as "lifelike" as possible. It's worth remarking, though, that many of the basic tools and methods of computer graphics draw directly on the very ideas we'll develop in this book.

Though we'll sometimes mention computer graphics.

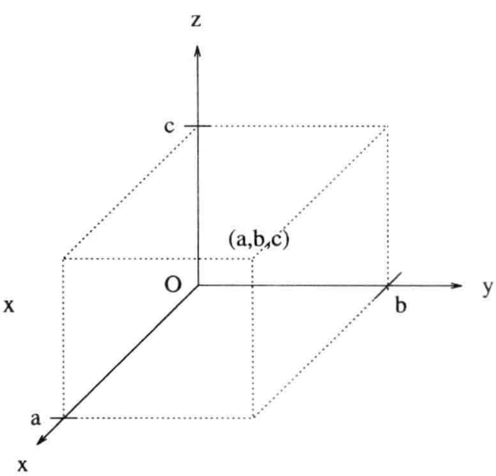
Cartesian coordinates in three dimensions

The *idea* of Cartesian coordinates is the same in both two and three dimensions, but the pictures look a little different. Compare these:

Coordinates in 2-d



Coordinates in 3-d



We'll occasionally use other axis labels.

Recall the formalities in the xy-plane. A **Cartesian coordinate system** consists of an **origin**, labeled O, and horizontal and vertical **coordinate axes**, labeled x and y, passing through O. On each axis we choose a positive direction ("east" and "north", usually) and a unit of measurement (not necessarily the same on both axes).

Given such a coordinate system, every point P in the plane corresponds to one and only one ordered pair (a, b) of real numbers, called the **Cartesian coordinates** of P. The pair (a, b) can be thought of as P's "Cartesian address": To reach P from the origin, move a units in the positive x-direction and b units in the positive y-direction.

If a or b is negative, go the other way.

There's "room" in $\mathbb{R}^3$ for three mutually perpendicular axes. $\mathbb{R}^2$ has room for only two.

Coordinates in three-dimensional xyz-space work the same way—but with *three* coordinate axes, labeled x, y, and z. Each axis is perpendicular to the other two. To reach the point $P(a, b, c)$ from the origin, go a units in the positive x-direction, b units in the positive y-direction, and c units in the positive z-direction. As the figure above illustrates, the resulting point $P(a, b, c)$ can also be thought of as a corner (the one opposite to the origin) of a rectangular solid with dimensions $|a|$, $|b|$, and $|c|$.

Quadrants and octants. The two axes divide the xy-plane into four **quadrants**, defined by the pattern of positive or negative x- and y-coordinates. The analogous regions in xyz-space are called **octants**. The **first octant**, for instance, consists of all points (x, y, z) with all three coordinates positive. In the picture above, only the first octant is visible. The picture below gives another view. There are *eight* octants in all in xyz-space—one for each of the possible patterns $(+, +, +)$, $(-, +, +)$, ..., $(-, -, -)$ of signs of the three coordinates.

As the name suggests.

Are you certain that 8 is the right number? See the exercises for more.

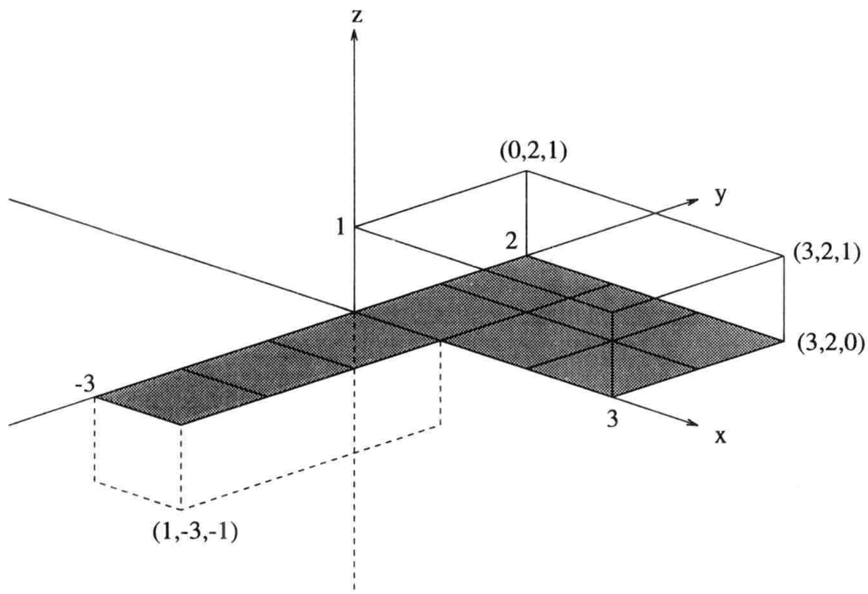
Coordinate planes. One can think of the first octant as a room, with the origin at the lower left corner of the front wall. At this point, three walls meet, all at right angles. These "walls" are known as the **coordinate planes**: the yz-plane (the front wall), the xy-plane (the floor), and the xz-plane (the left wall). These coordinate planes correspond to simple equations in the variables x, y, and z. The yz-plane, for example, is the graph of the equation $x = 0$, i.e., the set of all points (x, y, z) that satisfy this equation. The xy- and xz-planes, similarly, are graphs of the equations $z = 0$ and $y = 0$, respectively.

Be sure you agree: look carefully at the standard picture.

Many possible views. The xy-plane, being "flat," is relatively easy to draw. Simulating three-dimensional space on a flat page or computer screen is, of course, much harder, and there is always some price to be paid in distortion. For instance, the x-, y-, and z-axes are, in 3-d reality, all perpendicular to each other, but no flat picture can really show this. The axes shown below, for instance, don't make right angles on the page:

Study this picture carefully; it's worth the effort.

Three-Dimensional Coordinates



The picture's view of xyz-space is somewhat different from that above. Observe:

Horizontal and vertical. The xy-plane (in which the shaded "floor tiles" lie) is drawn to appear *horizontal*. The z-axis is *vertical*; the positive direction is up. This is a standard convention; we'll follow it consistently.

Hidden lines. The dashed lines in the picture lie "below" the xy-plane. They would be hidden from view if the xy-plane—the "floor"—were opaque. How much of xyz-space is considered visible is a matter of choice. Sometimes, only the first octant is shown.

Positive directions. An arrow on each axis indicates the positive direction. In particular, the 3×2 block of shaded squares lies in the first quadrant of the xy-plane. The other shaded squares lie in the plane's fourth quadrant.

Plotting points: positive and negative coordinates. Any point $P(a, b, c)$ is plotted the same way: from the origin move a, b, and c units in the positive x-, y-, and z-directions, respectively. Negative coordinates cause no special problem—just move the other way.

Where's the viewer? The picture is drawn as though the viewer were floating somewhere above the *fourth quadrant* of the xy-plane. In the previous 3-d picture, by contrast, the viewer floats somewhere above the *first* quadrant.

Think about this. Do you agree?

There's nothing sacred about either viewing angle; we'll use various viewpoints as we go along. So, for that matter, do the various computer plotting packages readers may have at hand.

No perspective. To a human viewer, rectangular boxes like those above would appear "in perspective"; the sides would seem to taper toward a vanishing point. The closer the viewer, the more pronounced these effects would be. For the sake of simplicity, we ignore perspective effects in the picture above, (and elsewhere in this book). In effect, the viewer is assumed to be very, very far from the origin, perhaps looking through a telescope.

There is no single "best" picture of a 3-d object; choosing a good or convenient view may depend on properties of the object, what needs emphasis, or even on the drawing technology at hand.

Computer, calculator, pencil, sharp stick, ...

Distance and midpoints

Let $P(x_1, y_1)$ and $Q(x_2, y_2)$ be any two points in the xy-plane. Recall that the **distance** from P to Q is given by the familiar "Pythagorean" formula

Or from Q to P—it doesn't matter.

$$d(P, Q) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2},$$

and that the **midpoint** M of the segment joining P to Q has these "averaged" coordinates:

$$M = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right).$$

The formulas in three dimensions aren't much different:

Definition: The **distance** between $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ is

$$d(P, Q) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}.$$

The **midpoint** of the segment joining P and Q has coordinates

$$M\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2}\right).$$

Both definitions are simply three-dimensional variations of the corresponding formulas in the xy-plane. In both two and three dimensions, for example, distance is computed as the square root of the sum of the squared differences in coordinates.

In either two or three dimensions, the distance formula reflects the Pythagorean rule. See the exercises for more details.

■ **Example 1.** Consider the points $P(0, 0, 0)$ and $Q(2, 4, 6)$. Find the distance from P to Q and the midpoint M of the segment joining them. How far is M from P and from Q?

Solution: By the distance formula,

$$d(P, Q) = \sqrt{(2 - 0)^2 + (4 - 0)^2 + (6 - 0)^2} = \sqrt{56} \approx 7.483.$$

The midpoint, according to the formula, is $M(1, 2, 3)$—each coordinate of M splits the difference between the corresponding coordinates of P and Q. To see why M deserves the name "midpoint," notice that

$$\begin{aligned} d(P, M) &= \sqrt{(1 - 0)^2 + (2 - 0)^2 + (3 - 0)^2} \\ &= \sqrt{(2 - 1)^2 + (4 - 2)^2 + (6 - 3)^2} = \sqrt{14} \approx 3.742. \end{aligned}$$

Thus M lies—as a midpoint should—halfway between P and Q. □

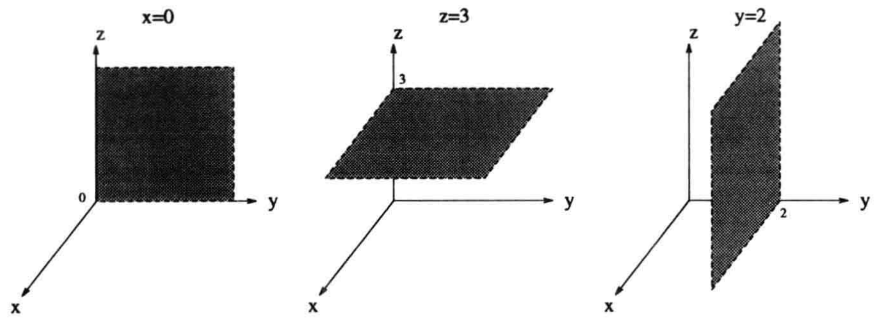
Equations and their graphs

The graph of an equation in x and y is the set of all points (x, y) that satisfy the equation. The graph of $x^2 + y^2 = 1$, for instance, is a circle of radius one in the xy-plane, centered at the origin. The graph of the equation $x = 0$ is the y-axis.

The graph of an equation may or may not be the graph of a function. The unit circle is not a function graph.

The same idea applies for three variables: the graph of an equation in x, y, and z is the set of points (x, y, z) in space that satisfy the equation. Here, graphically, are three simple examples (only the first octant is shown):

Three simple graphs



Solutions of the equation $x = 0$ are points of the form $(0, y, z)$, so the graph is the yz-plane. Similarly, solutions of $z = 3$ are all points of the form $(x, y, 3)$, so the graph is a horizontal plane, floating 3 units above the xy-plane. The graph of $y = 2$ is parallel to the xz-plane, but moved two units in the positive y-direction.

Notice that in each case above, the graph of an equation in x, y, and z is a plane—a *two-dimensional* object. By comparison, the graph of one equation in x and y is usually a curve or a line—a *one-dimensional* object. The pattern is the same in both cases: the graph of an equation has dimension *one less than* the number of variables.

Watch for this pattern as we go along.

A few special types of graphs in xyz-space deserve special mention.

Planes

A **linear equation** is one of the form $ax + by + cz = d$, where a, b, c, and d are constants, with at least one of a, b, and c nonzero. All three equations plotted above are linear; they illustrate an important general fact:

What goes wrong if $a = b = c = 0$?

The graph of any linear equation is a plane.

(We'll explain this fact carefully in a later section.) To draw planes in xyz-space, we'll use the the fact that a plane is uniquely determined by three points (unless the points happen to be on a straight line).

■ **Example 2.** Plot the linear equation $x + 2y + 3z = 3$ in the first octant.

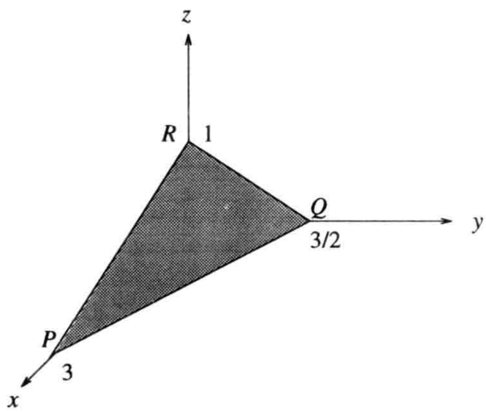
Solution: First we'll find some points (x, y, z) that satisfy $x + 2y + 3z = 3$. This is, if anything, too easy—there are infinitely many possibilities. Given *any* values for x and y, the equation determines a corresponding value for z. If, say, $x = 1$ and $y = 1$, then $x + 2y + 3z = 3$ can hold only if $z = 0$. Similarly, setting $y = 2$ and $z = 3$ forces $x = -10$. Among all possible solutions, here are three of the simplest:

Check this calculation.

$$P(3, 0, 0); \quad Q(0, 3/2, 0); \quad R(0, 0, 1).$$

These solutions are both easy to find (set two coordinates to zero and solve for the third) and easy to plot (they lie on the coordinate axes). Now we can plot our plane:

A plane in the first octant: x+2y+3z=3



Notice:

Intercepts. A typical line in the xy-plane has x- and **y-intercepts**, where the line intersects the coordinate axes. In a similar sense, a typical plane in xyz-space has x-, y-, and z-intercepts. In the picture, the intercepts are P, Q, and R.

Not every line in the xy-plane intercepts both axes; not every plane in space intercepts all three axes. See the exercises for more on this.

Traces. If we "slice" a surface in xyz-space with a plane, the intersection of the surface with the plane is called the **trace** of the surface in that plane. The plane p shown above meets each of the three coordinate planes in a straight line. These three lines, therefore, are the traces of the surface $x+2y+3z = 3$ in the xy-plane, the xz-plane, and the yz-plane, respectively.

Different planes give different traces.

See the picture.

It's easy to find equations for these traces. For example, a point (x, y, z) lies both in the plane p *and* in the xy-plane if and only if satisfies both $x + 2y + 3z = 3$ *and* $z = 0$. Setting $z = 0$ in the first equation gives $x + 2y = 3$—as expected, the equation of a line in the xy-plane. This line, therefore, is the trace of p in the xy-plane.

Do you see this line in the picture?

□

Spheres

In the plane, a circle of radius $r > 0$, with center $C(a, b)$ is the set of points $P(x, y)$ at distance r from (a, b). Translating this description into symbolic language produces the familiar formula for a circle in the plane:

$$d(P, C) = \sqrt{(x-a)^2 + (y-b)^2} = r, \quad \text{or} \quad (x-a)^2 + (y-b)^2 = r^2.$$

(Squaring both sides does no harm, and simplifies the equation's appearance.)

The analogous object in space to a circle in the plane is a **sphere** of radius r. Like a circle, a sphere is "hollow," similar to the skin of an orange. Adding the interior (the edible part of the orange) produces a **ball**. Like a circle, a sphere is the set of points at some fixed distance—the radius—from a fixed center point. Given a radius $r > 0$ and a center point $C(a, b, c)$, the sphere of radius r, centered at C, is the set of points (x, y, z) such that

$$d(P, C) = \sqrt{(x-a)^2 + (y-b)^2 + (z-c)^2} = r,$$
$$\text{or} \quad (x-a)^2 + (y-b)^2 + (z-c)^2 = r^2.$$

The simplest example, the **unit sphere**, has center (0, 0, 0) and radius 1; its equation reduces to this simple form:

$$x^2 + y^2 + z^2 = 1.$$

Drawing circles in the xy-plane is easy, even by hand. Drawing spheres (or *any* "curved" objects, for that matter) convincingly by hand is much harder.➻ Fortunately, rough sketches usually suffice.

For starters, circles in space don't always look circular. Depending on the viewing angle, they may look like ellipses.

Completing the square may reveal an equation's spherical form.

■ **Example 3.** Is the graph of $x^2 - 2x + y^2 - 4y + z^2 - 6z = 0$ a sphere? Which one?

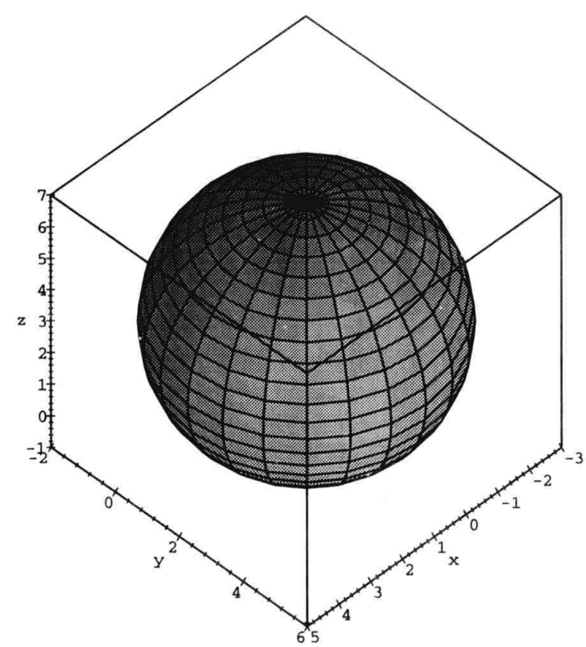
Solution: Completing the square in each variable separately gives➻

Check each step.

$$\begin{aligned} x^2 - 2x + y^2 - 4y + z^2 - 6z &= 0 \iff \\ (x^2 - 2x + 1) + (y^2 - 4y + 4) + (z^2 - 6z + 9) &= 1 + 4 + 9 \iff \\ (x-1)^2 + (y-2)^2 + (z-3)^2 &= 14. \end{aligned}$$

The last form shows that our equation describes the sphere of radius $\sqrt{14}$, centered at $(1, 2, 3)$. As the equation shows, this sphere passes through the origin. The picture shows this, too:

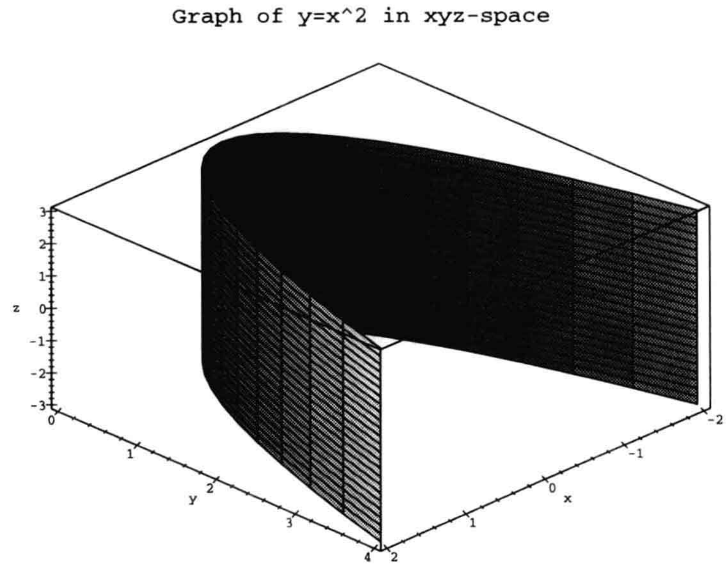
Graph of x^2-2x+y^2-4*y+z^2-6z=0



Cylinders

What is the graph of the equation $y = x^2$? The answer depends on where we're working. In the xy-plane, the graph is the familiar parabola—all points of the form

(x, x^2). Here's the graph of the same equation in xyz-space:



Notice:

In other words, the graph has "vertical walls."

A missing variable. The graph is *unrestricted in the z-direction*—it contains all points that lie directly above or below the graph of $y = x^2$ in the xy-plane.

The graph has this property, of course, because z is "missing" in the equation $y = x^2$. This means that if (x, y) satisfies the equation, then so does *every* point (x, y, z)—regardless of the value of z.

"Easy" is a relative thing—drawing anything in xyz-space poses certain challenges.

What's a cylinder? Graphs like this one, in which one (or more) of the variables is unrestricted, are called **cylinders**. Any equation that omits one or more variables—$y = z$, say—has a cylindrical graph. Plotting cylinders is comparatively easy. If the equation involves only y and z, for instance, we first plot the equation in the yz-plane, and then "extend" the graph in the x-direction.

In everyday speech, "cylinder" usually means a circular tube. As the next example illustrates, the mathematical idea of a cylinder is much more general.

■ **Example 4.** Discuss the graph in xyz-space of the equation $z = 2 + \sin y$. Interpret the result as a cylinder.

The surface, like many graphs, continues forever; a picture shows only part of the graph.

Solution: There's no variable x in the equation, so the graph is unrestricted in the x-direction—i.e., it's a cylinder in x. Here's a representative view: