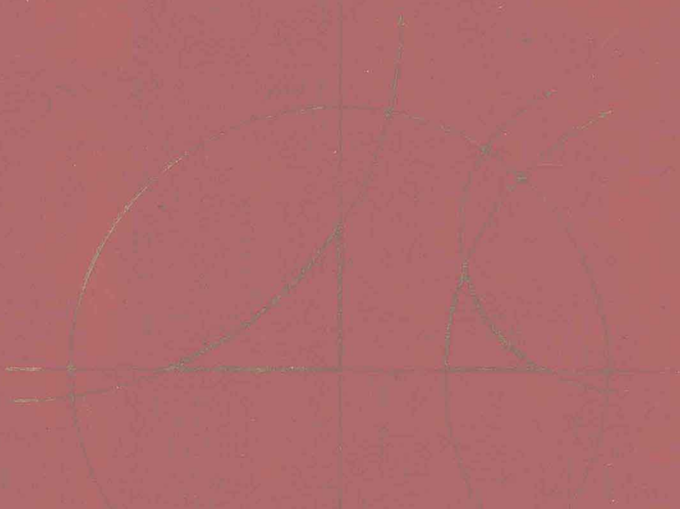
ROADS TO GEOMETRY



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PREFACE

The goal of this book is to provide a geometric experience which clarifies, extends, and unifies concepts generally discussed in traditional high school geometry courses and to present additional topics that assist in gaining a better understanding of elementary geometry. As its title *Roads to Geometry* indicates, this book is designed to provide the reader with a "map" for a voyage through plane geometry and its various branches. As prerequisites, this book assumes only a prior course in high school geometry and the mathematical maturity usually provided by a semester of calculus or discrete mathematics.

Preparations for this voyage begin in Chapter 1 with a discussion of the "Rules of the Road" in which the reader is familiarized with the properties of axiomatic systems and application of the axiomatic method to investigations of these systems. A discussion of several examples of finite and incidence geometries provides a framework within which we may investigate plane geometry.

With these preparations complete, the voyage commences in Chapter 2 where we are confronted with "Many Ways to Go." Here, within a historical perspective, we travel a variety of "roads" through geometry by investigating different axiomatic approaches to the study of Euclidean plane geometry. Axiomatic developments of geometry as proposed by Euclid, David Hilbert, G. D. Birkhoff, and the School Mathematics Study Group (SMSG) are compared and contrasted.

In Chapter 3, "Traveling Together," we investigate the content of neutral geometry. The SMSG postulates provide our pedagogical choice for a "main arterial" as we prepare ourselves for the choice between the Euclidean and non-Euclidean "exits" that appear on the horizon.

Chapter 4 provides "One Way to Go" as we explore the Euclidean plane. In this chapter we extend ideas developed in neutral geometry and provide a traditional look at the geometric topics of congruence, area, similarity, circles, and constructions from a Euclidean perspective.

While still within the Euclidean plane, Chapter 5 provides two "Side Trips" through analytical and transformational approaches to geometry. The real numbers, algebra, isometries, similarities, analytical transformations, and inversion and their applications to geometric theorem proving are discussed.

In Chapter 6 we consider "Other Ways to Go." We return briefly to neutral geometry in preparation for our venture into the non-Euclidean plane. In addition to a discussion of hyperbolic geometry, this chapter contains a detailed description of the Poincaré disk model and a brief excursion into elliptic geometry.

Finally, in Chapter 7, "All Roads Lead To . . ." projective geometry. Here we delve into a more general geometry than we have studied in previous chapters as we investigate the real projective plane and the ideas of duality, perspectivity, and projective transformations.

This text is appropriate for several kinds of students. Preservice teachers of geometry are provided with a rigorous yet accessible treatment of plane geometry in a historical context. Mathematics majors will find its axiomatic development sufficiently rigorous to provide a foundation for further study in the areas of Euclidean and non-Euclidean geometry. Through the choice of the SMSG postulate set as a basis for the development of plane geometry, this book avoids the pitfalls of many "foundations of geometry" texts which encumber the reader with such a detailed development of preliminary results that many other substantive and elegant results are inaccessible in a one-semester course.

The chapters of this book separate nicely into independent units. The material in Chapters 1 and 2 provides preliminary groundwork for the study of geometry. Instructors who feel that their classes are exceptionally well prepared can omit these chapters in the interest of freeing time for material presented later in the book. Instructors teaching more typical classes will find the discussion of axiomatics in Chapter 1 and the comparisons of the various axiom sets in Chapter 2 very helpful in conveying the notion of mathematical rigor. Instructors can teach a semester of Euclidean geometry using Chapters 1 through 5, while those instructors more interested in non-Euclidean Geometries can opt to cover Chapters 1, 2, 3, and 6.

At the end of each section is an ample collection of exercises of varying difficulty which provide problems that both extend and clarify results of the section as well as problems that apply those results. At the end of each of Chapters 3 through 7 is a summary listing all the new definitions and theorems of the chapter. In addition, a "tear-out" page listing the SMSG axioms is included in the back cover so that the student does not have to turn to an appendix each time an axiom is invoked.

The authors hope that *Roads to Geometry* will in some way encourage the reader to more fully appreciate the marvelous worlds of Euclidean and non-Euclidean plane geometry and to that end we wish you bon voyage.

E. C. Wallace
S. F. West

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The remainder of this section briefly introduces several of the great philosopher/mathematicians of antiquity and their roles in the birth of "demonstrative geometry."

Thales of Miletus

The transformation of the study of geometry from a purely practical science (namely, surveying) to a branch of pure mathematics was undertaken by Greek scholars and took place over a number of centuries. The individual most often credited with initiating the formal study of demonstrative geometry as a discipline is Thales of Miletus (c. 640–546 B.C.). In his early days Thales was a merchant, and in this capacity he traveled to Egypt and the Middle East. He returned to Greece with a knowledge of the measurement techniques used by the Egyptians at that time. The greatest contribution made by Thales to the study of geometry was his ability to abstract the ideas of the Egyptians from a physical context to a mental one. The propositions for which he has been given credit are among the simplest in plane geometry. For example, Proclus, in his "Eudemian Summary,"[2] stated that Thales was "the first to demonstrate that the circle is bisected by the diameter."[3] As one can see, this rather simple assertion is not noteworthy by virtue of its profound content. The significance of Thales' contribution lies not in the content of the propositions themselves but in his use of logical reasoning to argue in favor of them. Proclus describes an indirect "proof" (presumably due to Thales) to support the circle bisection theorem. While Thales' proof is not acceptable by today's standards (and was even avoided by Euclid) it showed, for the first time, an attempt to justify geometric statements using reason instead of intuition and experimentation.

D. E. Smith speaks about the importance of Thales' work in geometry: "Without Thales there would not have been a Pythagoras—or such a Pythagoras; and without Pythagoras there would not have been a Plato—or such a Plato."[4]

Pythagoras

Pythagoras (c. 572 B.C.) was born on the Greek Island of Samos before Thales' death and was probably a student of Thales. Pythagoras traveled widely throughout the Mediterranean region, and it is very possible that his

[2] The "Eudemian Summary" is a small part of Proclus' *Commentary on the First Book of Euclid* in which he describes the history of Thales as given by Eudemus of Rhomes in a work available to Proclus but which has since been lost to us.

[3] Glenn R. Morrow, (trans.), *Proclus—A Commentary on the First Book of Euclid's Elements* (Princeton, N.J.: Princeton University Press, 1970), p. 124.

[4] D. E. Smith, *History of Mathematics* (New York: Dover Publications, Inc., 1958), I, p. 68.

journeys took him to India, since his philosophical orientation was more closely aligned with the Indian civilization than with the Greek. On returning to Europe, Pythagoras migrated to Croton, a Greek colony in southern Italy, and established a quasi-religious brotherhood called the Pythagoreans. It is likely that much of the mathematics attributed to Pythagoras was actually developed by members of this brotherhood during the 200 or so years of its existence. The Pythagoreans took mathematical thought a step beyond the point to which Thales had brought it. Whereas Thales had formalized a portion of the geometry that he encountered, the Pythagorean philosophy was to develop mathematical results exclusively as the result of deduction. It was during this time that "chains of propositions were developed in which each successive proposition was derived from earlier ones."[5] The Pythagorean school set the tone for all the Greek mathematics that was to follow, and since the ideas of Plato were largely committed to mathematics, one could say that Pythagoras had a major effect on all of Greek philosophy.

Plato

Plato's role in the development of geometry (and of mathematics in general) is often overshadowed by his preeminent status in Greek philosophy in general. The Academy of Plato, which was established about 387 B.C., attracted the most famous scholars of the time. At the Academy the study of mathematics was confined to pure mathematics, with the emphasis placed on soundness of reason. One of Plato's most famous students was Aristotle (c. 584 B.C.), who in his work *Analytica Posteriora* did much to systematize the classical logic that formed a basis for all Greek mathematics. By about 400 B.C. Greek civilization had developed to the point where intellectual pursuits were valued for their intrinsic virtue. Plato was of the mind that "mathematics purifies and elevates the soul."[6] Since there was no need for mathematicians at the Academy to concern themselves with applications of their work, the emphasis could be placed on the processes involved in the development of mathematical thought rather than on worldly products of that thought. Thus mathematics had by 350 B.C. taken on the nature of a "pure science."

Euclid

The name most often associated with ancient Greek geometry is that of Euclid. Not a great deal is known about Euclid's background. He may have been born in Greece, or he may have been an Egyptian who went to Alexan-

[5] Eves, in *Historical Topics*, p. 172.

[6] Morrow, *Proclus—A Commentary*, p. 25.

dria to study and teach. He is believed to have been the first mathematics professor at the great University of Alexandria. His lifetime overlapped that of Plato, and he may have been a student at Plato's academy. Proclus, in his *Commentary*, tells us that Euclid was influenced by Plato's philosophy, but there is no direct evidence that the two ever met.

By Euclid's time (c. 325 B.C.) the development of rational thought had progressed sufficiently to allow for, and even demand, a systematic study of geometry. Euclid's monumental work, *Elements of Geometry*, a single chain of 465 propositions which in part encompasses plane and solid geometry, has for over 2000 years, remained as the most widely known example of a formal axiomatic system. As we will see in subsequent chapters, Euclid's work was far from flawless. Still its strengths far outnumber its weaknesses, as attested by the fact that it overshadowed and replaced all previous writings in this area.

In order to place Euclid's historic effort in context, we shall in the next section discuss what is meant by an axiomatic system and investigate the properties that axiomatic systems possess.

EXERCISE SET 1.1

1. As indicated earlier, Egyptian geometers used the formula $A = \frac{1}{4}(a + c)(b + d)$ to calculate the area of any quadrilateral whose successive sides have lengths a, b, c, and d.
   (a) Does this formula work for squares? For rectangles that are not squares?
   (b) If you choose specific lengths for the sides of an isosceles trapezoid, how does the result compare to the actual area? Repeat for two other isosceles trapezoids. Do the same for three specific parallelograms.
   (c) Generalize your results for part (b).
2. If a and b are the lengths of the legs of a right triangle and c is the length of the hypotenuse, Babylonian geometers approximated the length of the hypotenuse by the formula $c = b + (a^2/2b)$.
   (a) How does this approximation compare to the actual result when $a = 3$ and $b = 4$? When $a = 5$ and $b = 12$? When $a = 12$ and $b = 5$?
   (b) Give an algebraic argument demonstrating that this formula results in an approximation that is too large.
3. The following was translated from a Babylonian tablet created about 2600 B.C. Explain what it means.

> 60 is the circumference, 2 is the perpendicular, find the chord. Double 2 and get 4, do you see? Take 4 from 20 and get 16. Square 20, and you get 400. Square 16, and you get 256. Take 256 from 400 and you get 144. Find the square root of 144. 12, the square root, is the chord. This is the procedure.[7]

[7] Howard Eves, *A Survey of Geometry* (Boston: Allyn and Bacon, 1965), p. 7, Problem 1.2-2.

4. The Moscow Papyrus (c. 1850 B.C.) contains the following problem:

> If you are told: A truncated pyramid of 6 for the vertical height by 4 on the base by 2 on the top. You are to square this 4, result 16. You are to double 4, result 8. You are to square 2, result 4. You are to add the 16, the 8, and the 4, result 28. You are to take one third of 6, result 2. You are to take 28 twice, result 56. See, it is 56. You will find it right.

Show that this is a special case of the general formula, $V = \frac{1}{3}h(a^2 + ab + b^2)$, for the volume of the frustum of a pyramid whose bases are squares, whose sides are a and b, respectively, and whose height is h.

5. An Egyptian document, the Rhind Papyrus (c. 1650 B.C.), states that the area of a circle can be determined by finding the area of a square whose side is $\frac{8}{9}$ of the diameter of the circle. Is this correct? What value of π is implied by this technique?
6. It is said that Thales indirectly measured the distance from a point on shore to a ship at sea using the equivalent of angle-side-angle (ASA) triangle congruence theorem. Make a diagram that could be used to accomplish this feat.
7. Eratosthenes (c. 275 B.C.), a scholar and librarian at the University at Alexandria, is credited with calculating the circumference of the earth using the following method: Eratosthenes observed that on the summer solstice the sun was directly overhead at noon in Syene (the present site of Aswan), while at the same time in Alexandria, which was due north, the rays of the sun were inclined 7°12′, thus indicating that Alexandria was 7°12′ north of Syene along the earth's surface. Using the known distance between the two cities of 5000 stades (approximately 530 miles), he was able to approximate the circumference of the earth. Make a diagram that depicts this method and calculate the circumference in stades and in miles. How does this result compare to present-day estimates?

1.2 AXIOMATIC SYSTEMS AND THEIR PROPERTIES

The Axiomatic Method

As we begin our study of geometry, it is important that we have a basic understanding of the *axiomatic method* used in the development of all of modern mathematics. The axiomatic method is a procedure by which we demonstrate or prove that results (theorems, and so on) discovered by experimentation, observation, trial and error, or even by "intuitive insight," are indeed correct. Little is known about the origins of the axiomatic method. Most historians, relying on accounts given by Proclus in his "Eudemian Summary," indicate that the method seems to have begun its evolution during the time of the Pythagoreans as a further development and refinement of various early deductive procedures.

In an axiomatic system the proof of a specific result is simply a sequence of statements, each of which follows logically from the ones before and leads from a statement that is known to be true to the statement which is to be proven. First, for a proof to be convincing, it is necessary to establish ground rules for determining when one statement follows logically from another. For the purposes of this development, our rules of logic will consist of the standard two-valued logic studied in most introductory logic courses. Second, it is important that all readers of the proof have a clear understanding of the terms and statements used in the discussion. To ensure this clarity, we might try to define each of the terms in our discussion. If, however, one of our definitions contains an unfamiliar term, then the reader has the right to expect a definition of this term. Thus a chain of definitions is created. This chain must be circular or linear (think of a concrete model). Since circularity is unacceptable in any logical development, we may assume that the chain of definitions is linear. Now this linear chain may be an infinite sequence of definitions, or it must stop at some point. An unending sequence of definitions is at best unsatisfying, so the collection of definitions must end at some point and one or more of the terms will remain undefined. These terms are known as the *undefined* or *primitive terms* of our axiomatic system.

The primitive terms and definitions can now be combined into the statements or *theorems* of our axiomatic system. For these theorems to be of mathematical value, we must supply logically deduced proofs of their validity. We now need additional statements to prove these theorems which in turn require proof. As before, we form a chain of statements that leads us to the conclusion that, to avoid circularity, one or more of these statements must remain unproven. These statements, called *axioms* or *postulates,*[8] must be assumed, and they form the fundamental truths[9] of our axiomatic system.

To summarize, any logical development of an axiomatic system must therefore conform to the pattern represented in Table 1.2.1.

To illustrate an axiomatic system and the relationships among its undefined terms, axioms, and theorems we will consider the following example.

Example 1.2.1

A simple abstract axiomatic system. Undefined terms: Fe's, Fo's, and the relation, "belongs to."[10]

[8] Today, the words "axiom" and "postulate" are used interchangeably. Historically, the word "postulate" has been used to represent an assumed truth confined to a particular subject area, while "axiom" represents a more universal truth applicable to all areas of mathematics.

[9] The truth of these axioms is not at issue—just the reader's willingness to accept them as true.

[10] We will occasionally use the terminology "a Fe is on a Fo" or "a Fo contains a Fe," and by this we mean that the Fe "belongs to" the Fo.

TABLE 1.2.1 The Axiomatic Method

1. Any axiomatic system must contain a set of technical terms that are deliberately chosen as undefined terms and are subject to the interpretation of the reader.
2. All other technical terms of the system are ultimately defined by means of the undefined terms. These terms are the definitions of the system.
3. The axiomatic system contains a set of statements, dealing with undefined terms and definitions, that are chosen to remain unproven. These are the axioms of the system.
4. All other statements of the system must be logical consequences[11] of the axioms. These derived statements are called the theorems of the axiomatic system.

AXIOM 1. There exist exactly three distinct Fe's in this system.

AXIOM 2. Two distinct Fe's belong to exactly one Fo.

AXIOM 3. Not all Fe's belong to the same Fo.

AXIOM 4. Any two distinct Fo's contain at least one Fe which belongs to both.

FE-FO THEOREM 1. Two distinct Fo's contain exactly one Fe.

Proof. Since Axiom 4 states that two distinct Fo's contain at least one Fe, we need only show that these two Fo's contain no more than one Fe. For this purpose we will use an indirect proof and assume that two Fo's share more than one Fe. The simplest case of "more than one" is two. Now each of these two Fe's belong to two distinct Fo's, but that in turn contradicts Axiom 2, and we are done.

FE-FO THEOREM 2. There are exactly three Fo's.

Proof. Axiom 2 tells us that each pair of Fe's is on exactly one Fo. Axiom 1 provides us with exactly three Fe's, and by counting distinct pairs of Fe's, we find that we have at least three Fo's. Now suppose that there exists a distinct fourth Fo. Theorem 1 tells us that the fourth Fo must share a Fe with each of the other Fo's. Therefore it must contain at least one of the two of the existing three Fe's, but Axiom 2 prohibits this. Therefore a fourth Fo cannot exist, and there are exactly three Fo's.

[11] Recall that it is presumed that underlying the axiomatic system is some type of logical structure on which valid arguments are based.