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Jin Akiyama
Edy Tri Baskoro
Mikio Kano (Eds.)

Combinatorial Geometry and Graph Theory

Indonesia-Japan Joint Conference, IJCCGGT 2003
Bandung, Indonesia, September 2003
Revised Selected Papers

Springer

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Preface

This volume consists of the refereed papers presented at the Indonesia-Japan Joint Conference on Combinatorial Geometry and Graph Theory (IJCCGGT 2003), held on September 13–16, 2003 at ITB, Bandung, Indonesia. This conference can also be considered as a series of the Japan Conference on Discrete and Computational Geometry (JCDCG), which has been held annually since 1997. The first five conferences of the series were held in Tokyo, Japan, the sixth in Manila, the Philippines, in 2001, and the seventh in Tokyo, Japan in 2002.

The proceedings of JCDCG 1998, JCDCG 2000 and JCDCG 2002 were published by Springer as part of the series Lecture Notes in Computer Science: LNCS volumes 1763, 2098 and 2866, respectively. The proceedings of JCDCG 2001 were also published by Springer as a special issue of the journal *Graphs and Combinatorics*, Vol. 18, No. 4, 2002.

The organizers are grateful to the Department of Mathematics, Institut Teknologi Bandung (ITB) and Tokai University for sponsoring the conference. We also thank all program committee members and referees for their excellent work. Our big thanks to the principal speakers: Hajo Broersma, Mikio Kano, Janos Pach and Jorge Urrutia. Finally, our thanks also goes to all our colleagues who worked hard to make the conference enjoyable and successful.

August 2004

Jin Akiyama
Edy Tri Baskoro
Mikio Kano

Organization

The Indonesia-Japan Joint Conference on Combinatorial Geometry and Graph Theory (IJCCGGT) 2003 was organized by the Department of Mathematics, Institut Teknologi Bandung (ITB) Indonesia and RIED, Tokai University, Japan.

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Table of Contents

On Convex Developments of a Doubly-Covered Square

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Abstract. We give an algebraic characterization of all convex polygons that are 2-flat foldable to a square, that is, we determine all shapes of convex developments of a doubly-covered square.

1 Doubly-Covered Square

Let us introduce an equivalence relation on the plane $E = \{ (x, y) \mid x, y \in \boldsymbol{R}\}$ in the following way.

Definition 1. *We say that two points $X_1(x_1, y_1)$ and $X_2(x_2, y_2)$ of the plane E are equivalent if either one of the following two conditions is satisfied:*

(1) The points X_1 and X_2 are symmetric with respect to some lattice point. Namely, the midpoint of the line segment X_1X_2 is a lattice point.

(2) The point X_1 can be moved to the point X_2 by means of a parallel translation given by a vector whose components are both even integers. Namely, the components of the vector $\overrightarrow{X_1X_2}$ are both even integers.

The fact that the conditions above define an equivalence relation is obvious since the composition of any combinations of motions, moving points to those which are symmetric in the sense of (1) or parallel translations of the type (2), yields again the motion of the type (1) or (2).

Lemma 1. *Denote by P the quotient space obtained from E by means of the equivalence relation introduced in Definition 1, and denote by p the quotient map $E \to P$. Then, P can be identified with a doubly-covered square (Fig. 1).*

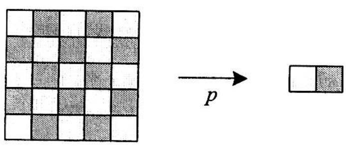

Fig. 1

J. Akiyama et al. (Eds.): IJCCGGT 2003, LNCS 3330, pp. 1–13, 2005.

Proof. Let us show first that a representative of an equivalence class for the given equivalence relation can be chosen in the set

$$R = \{ (x, y) \mid -1 \leq x \leq 1, 0 \leq y \leq 1 \}.$$

Letters k and l represent integers in the sequel.

(a) Any point (x, y) satisfying the conditions $2k-1 \leq x < 2k+1$, $2l-1 \leq y < 2l$ is equivalent to a point in R by means of a motion of the type (1) moving points in E to those which are symmetric with respect to the point (k, l).
(b) Any point (x, y) satisfying the conditions $2k-1 \leq x < 2k+1$, $2l \leq y < 2l+1$ is equivalent to a point in R by means of the parallel translation of the type (2) given by the vector $(-2k, -2l)$.

Therefore, we see that a representative of an equivalence class can be chosen in the set R. We next investigate equivalence of the points within R by means of the given equivalence relation. Let us represent the set R as the rectangle $ABB'A'$ as in Fig. 2. No point in Int(R), the interior of R, is equivalent to another point in R. On the other hand, there are pairs of equivalent points on ∂R, the boundary of R. Two points on the line segment AA' are equivalent if they are symmetric with respect to the point O. Similarly, two points on the line segment BB' are equivalent if they are symmetric with respect to the point C. Furthermore, a point on the line segment AB is equivalent to a point on the line segment $A'B'$, if they can be moved to each other by means of the parallel translation given by a vector of length 2 in the direction of the x-axis.

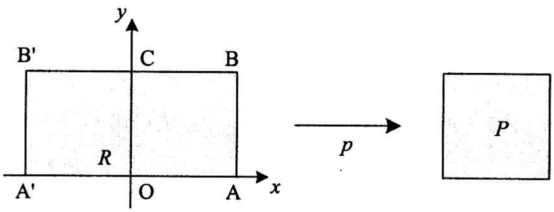

Fig. 2

□

From these observations we conclude that the quotient space P for the equivalence relation given in Definition 1 can be identified with the figure obtained by folding the rectangle R along the line segment OC, and gluing together segments AO and $A'O$, segments BC and $B'C$, and segments AB and $A'B'$. What results is a doubly-covered square.

The following two corollaries can be proved easily.

Corollary 1. *The set of all the lattice points in the plane E can be partitioned into four equivalence classes* {(odd, odd), (odd, even), (even, odd), (even, even)} *depending on the parities of the coordinates, and each of these equivalence classes corresponds by p to each of the four vertices of P, respectively.*

Corollary 2. *Let A and B be an arbitrary pair of lattice points in the plane E, and denote by S the set of all the lattice points lying on the line segment AB. Then $p(S)$ consists of exactly two of the vertices of P. Namely, only two among the four equivalence classes described in Corollary 1 can lie on a straight line.*

2 Developments

Let us consider next what we mean by a development of a doubly-covered square. We will give a definition of a development by using the quotient map $p : E \to P$ introduced in the preceding section. It should be obvious that this definition coincides with the usual definition of a development.

Definition 2. *When a polygon V in the plane E satisfies the following conditions (1) - (3), V is called a* development *of the doubly-covered square P (Fig. 3).*

(1) The map $p|_V : V \to P$, which is the restriction of p to V, is surjective.
(2) $p(\mathrm{Int}(V)) \cap p(\partial V) = \emptyset$.
(3) $p|_{\mathrm{Int}(V)} : \mathrm{Int}(V) \to P$, which is the restriction of p to $\mathrm{Int}(V)$, is injective.

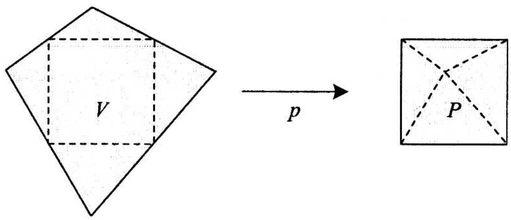

Fig. 3

The lines of folding involved in constructing P from a development V are the lines drawn in the plane E through lattice points parallel to the x- or y-axis.

Associated with this definition, we define as follows the *cut tree* which appears when a doubly-covered square P is developed into a development V.

Definition 3. *Let V be a development of a doubly-covered square P. We call the set $T = p(\partial V)$ the* cut tree *which appears when P is developed into V (Fig. 4).*

According to the results in [4] a cut tree has the following properties:

Lemma 2. *A cut tree T has the following properties:*

(1) T is a tree.
(2) T goes through every vertex of P.
(3) Leaves of T are the vertices of P.

From this lemma, we can get the following corollary easily.

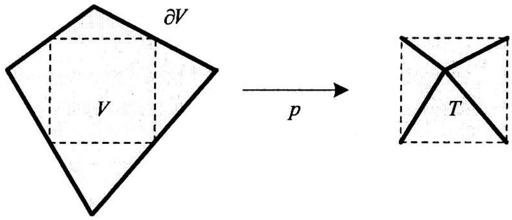

Fig. 4

Corollary 3. *Let V be a development of a doubly-covered square P. Then,*

(1) Int(V) *contains no lattice points in E.*
(2) For each vertex x of P, there exists at least one lattice point v in V such that $p(v) = x$.

The following lemma follows from Lemma 3.1(4) of paper [3].

Lemma 3. *In order for a point x to be a point of degree d of the cut tree T it is necessary and sufficient that the inverse image $(p|_{\partial V})^{-1}(x)$ of the point x under the map $p|_{\partial V} : \partial V \to P$ consists exactly of d points.*

As a special case of this lemma, we get the following corollary for leaves of a cut tree (points of degree 1):

Corollary 4. *Let a polygon V be a development of a doubly-covered square P with T as its cut tree. Let v be a lattice point lying in ∂V and let $x = p(v)$. Then, the following statements (1) - (3) are mutually equivalent:*

(1) x is a leaf of the cut tree T .
(2) The angle around v within the development V is $180°$.
(3) There exists no other point in ∂V which belongs to the same equivalence class as v.

Proof.
(1) $\Rightarrow$ (2): Let x be a leaf of T. Since the total angle around the vertex x within P is $180°$, it is clear that there is $180°$ angle around v within the development V.
(2) $\Rightarrow$ (3): If the total angle around the lattice point v within V is $180°$, then it is impossible to have a development which puts together more angles around v. If there exists another point v' in ∂V which is equivalent to v, then the angle $\theta > 0$ around the vertex v' must be added to the angle around the vertex x of P to make the total angle more than $180°$, which yields a contradiction. Therefore, there cannot be another point in ∂V which is equivalent to v.
(3) $\Rightarrow$ (1): If there exists no point in ∂V which is equivalent to v, then the inverse image $(p|_{\partial V})^{-1}(x)$ stated in Lemma 3(2) consists exactly of one point. Hence x must be a point of the cup tree T of degree 1, namely, it is a leaf of T. □

3 Faces of Parallelograms

The statement "doubly-covered square P has the top face and the bottom face" sounds plausible, but is it really true? To begin with, is it obvious that a doubly-covered square P consists of two faces? What are these two faces? Let us deal with these questions in this section.

Problem 1. Suppose we have a development $S = S_1 \cup S_2$ consisting of two unit squares sharing a common side as in Fig. 5, and suppose all of the vertices of the two squares S_1 and S_2 correspond under the quotient map p to the vertices of the doubly-covered square P, then we can decompose the doubly-covered square P into two congruent squares $p(S_1)$ and $p(S_2)$. Is such a decomposition unique?

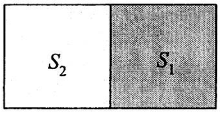

Fig. 5

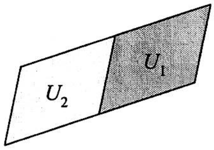

Fig. 6

In this problem, we assumed that all of the vertices of the two squares S_1 and S_2 correspond to the vertices of the doubly-covered square P under the map p. The reason for making this assumption is the fact that only four vertices of the doubly-covered square P have the property that the angle around each of those within the doubly-covered square P is only 180°. Every other point of P has the angle of 360° surrounding it in P.

The answer to Problem 1 above is "Yes, the decomposition is unique". We will call the decomposition $P = p(S_1) \cup p(S_2)$ given uniquely by the two squares the "congruent decomposition by squares". What about the following Problem 2? This is a question posed by replacing squares of Problem 1 by parallelograms.

Problem 2. Suppose we have as in Fig. 6, a development $U = U_1 \cup U_2$ consisting of two congruent parallelograms sharing a common side, and suppose all the vertices of the parallelograms U_1 and U_2 are mapped by the quotient map p onto the vertices of the doubly covered square P, then P can be decomposed into two faces $p(U_1)$ and $p(U_2)$, which are congruent parallelograms. How many such decompositions are there?

We call a decomposition $P = p(U_1) \cup p(U_2)$ into a pair of such congruent parallelograms a *congruent decomposition by parallelograms.* In answer to Problem 2 there are countable infinity of such decompositions. Fig. 7 and 8 illustrate examples of such decompositions. In these figures the diagram on the left gives a development U with two parallelograms U_1 and U_2 distinguished by different colors. The diagram on the right shows the result of constructing the doubly-covered square P by folding the development colored by the two different colors and then developing the result into a rectangle.

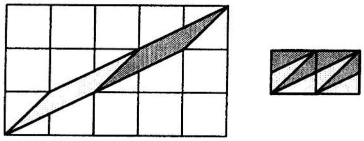

Fig. 7

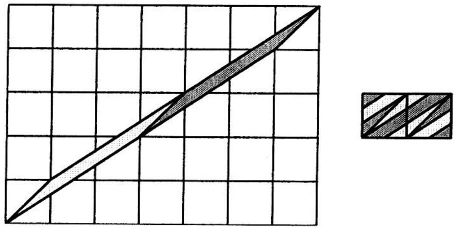

Fig. 8

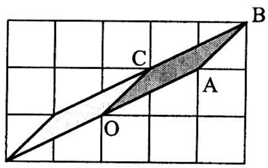

Fig. 9

How can we construct such congruent decompositions by parallelograms? Let us explain the situation by using Fig. 9. Let us first consider a parallelogram $OABC$ of area 1 where vertices are lattice points. (We orient the parallelogram so that $OABC$ refers to the labeling of the vertices in counter clock-wise direction, where O is the origin of the plane E). We will check later the fact that this parallelogram gives a development of a doubly-covered square.

Since the area of the parallelogram is 1, we know by Pick's Theorem that there are no other lattice points beside the vertices in the interior or on the sides of the parallelogram. Therefore, we can take an arbitrary pair of relatively prime integers (a, b) and let $\overrightarrow{OA} = (a, b)$. If the side OA is parallel to the x- or y-axis, then $(a, b) = (\pm 1, 0)$ or $(0, \pm 1)$, respectively.

For $\overrightarrow{OC} = (c, d)$, it suffices to find a pair (c, d) of integers satisfying $ad - bc = 1$ since the area of the parallelogram $OABC$ is equal to 1. Since a and b are relatively prime, we can find a pair c_0, d_0 satisfying $ad_0 - bc_0 = 1$ by using the extended Euclidean Algorithm. Using this pair (c_0, d_0), we let $(c, d) = (c_0 + ak, d_0 + bk)$ for an arbitrary integer k.

The procedure outlined above gives an explicit method for constructing a parallelogram of area 1 with all of its vertices on lattice points. In short, we can say that such a parallelogram can be determined for any choice of integers a, b, c, d satisfying the identity $ad - bc = 1$.

Let us denote by $SL(2, \boldsymbol{Z})$ the set of all 2×2 matrices with integer coefficients having determinant 1. We have the following lemma.

Lemma 4. *Denote by S_1 the unit square with vertices $O(0, 0)$, $A_1(1, 0)$, $B_1(1, 1)$, $C_1(0, 1)$. The following statements (1) and (2) concerning a parallelogram $OABC$ having the origin O as one of the vertices are mutually equivalent.*

(1) The parallelogram OABC has all its vertices on lattice points and has area 1.
(2) The parallelogram OABC is an image of S_1 under some linear transformation given by a matrix belonging to $SL(2, \mathbf{Z})$.

Proof. Suppose that the parallelogram $OABC$ has all of its vertices on lattice points and has area 1, and let $\overrightarrow{OA} = (a, b)$ and $\overrightarrow{OC} = (c, d)$. Then, a, b, c, d are all integers and $ad - bc = 1$. Therefore, if we let $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then the image of the unit square S_1 under the linear transformation f_M given by M is the parallelogram $OABC$. The converse assertion is obvious.

Next, we consider the situation indicated in Fig. 10. The diagram on the left indicates congruent unit squares S_1 and S_2, lying adjacently, while the diagram on the right shows two congruent parallelograms U_1 and U_2 of area 1 lying adjacently and sharing the side OC. By Lemma 4, it is clear that S_1 is mapped onto U_1 by means of a linear transformation f_M given by a matrix M belonging to $SL(2, \mathbf{Z})$. It is also obvious that S_2 is mapped onto U_2 by the same linear transformation f_M.

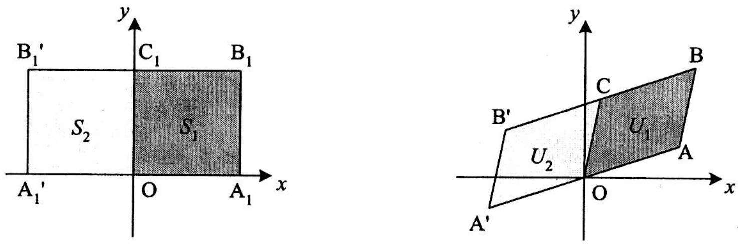

Fig. 10 □

In the next section we will show that the diagram on the right in Fig. 10 gives a development of the doubly-covered square P.

4 Actions of $SL(2, Z)$

The group $SL(2, \mathbf{Z})$ acts on the plane $E = \{ (x, y) \mid x, y \in \mathbf{R}\}$ as a group of linear transformations: $SL(2, \mathbf{Z}) \times E \to E$. The next lemma shows that this action preserves the equivalence relation on E given in Definition 1.

Lemma 5. *Let us denote by X_1, X_2 an arbitrary pair of points in E, and denote by $\sim$ the equivalence relation given in Definition 1. Then, for any $M \in SL(2, \mathbf{Z})$, $f_M(X_1) \sim f_M(X_2)$ is satisfied if $X_1 \sim X_2$.*

Proof. Suppose $X_1 \sim X_2$, then $X_1 X_2$ satisfy either the condition (1) or condition (2) of Definition 1. We will show that $f_M(X_1) \sim f_M(X_2)$ is satisfied in either case.